STUDENT'S REINVENTION OF STRAIGHT-LINE SOLUTIONS TO SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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This thesis is dedicated to Sakeenah for providing me with endless support and love during this adventure of spirit and mind.

The only secret in the universe is that there are no secrets.

-Author Unknown

ABSTRACT OF THE THESIS

Students' Reinvention of Straight-Line Solutions to Systems of
Linear Ordinary Differential Equations
by
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This paper analyzes how students who work in an inquiry-oriented classroom think and develop solution methods for analytic solutions to systems of two linear ordinary differential equations (ODEs). In particular, we seek to understand how students construct and interpret straight-line solutions (SLSs). SLSs are significant mathematical ideas because they serve as the basic building blocks for all other solutions. In particular, SLSs are eigensolutions that span the solution space. This report will also detail the role of proportional reasoning in the process of reinventing SLSs. Our theoretical orientation comes from a socio-constructivist view in which individual math activity is related to emerging classroom meanings and practices. Analysis of student work and justifications suggests that their mathematical understanding of SLSs is a dynamic and evolving entity that matures through classroom discussion and activity. Lastly, this study examines implications for teaching and for the revision of instructional materials.

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CHAPTER 1

INTRODUCTION

The broad purpose of this study is to analyze student thinking and solution methods for inventing analytic solutions to systems of two linear ordinary differential equations (ODEs). The first objective is to understand how students construct and interpret straight-line solutions (SLSs). SLSs are significant mathematical ideas because they serve as the basic building blocks for all other solutions[1]. A second objective of this study is to understand the extent to which students' methods of proportional reasoning contribute to their construction and use of SLSs. Students proportionally reason in many different mathematical settings from multiplication to calculus to solve proportion specific problems and to reason about other mathematical ideas (Kenny, Lindquist, & Hefferman, 2002; Lamon 1994). With systems of ODEs, students can use their prior knowledge about proportionality to invent their own methods for finding analytic solutions (Rasmussen & Keynes, 2003). The setting for this study is an inquiry-oriented classroom (Richards, 1991) in which students routinely explained and justified their thinking as they solved challenging problems for which they had to develop their own solution methods. Lastly, this study examines implications for teaching and for the revision of instructional materials.

Systems of ODEs naturally arise in the physical sciences and engineering and it is therefore important that both students and practitioners of these disciplines are familiar with them and their applications. One example of importance to structural engineering is the mathematical modeling of skyscrapers. The following is a system of non-linear ODEs that model the horizontal position and velocity of the skyscraper (Blanchard, Devaney & Hall, 1998).

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -x - y + x^3$$

[1] SLSs can be thought of as basis vectors that span the phase plane. The general solution to a system of linear ODEs is formed from a linear combination of the SLSs.

In this system *x* stands for the amount of horizontal displacement of the building from the rest position at any time *t* and *y* stands for the horizontal velocity of the building at any time *t*.

Non-linear systems such as this are pervasive in science. Unlike their linear counterparts, there are few analytic techniques available for solving non-linear systems because of their underlying mathematical complexity[2]. An important approach for analyzing their solutions is to approximate these systems with a linear system in a neighborhood around each equilibrium solution. The following is the linear approximation to the non-linear system near the equilibrium solution (1, 0):

$$\frac{du}{dt} = v$$
$$\frac{dv}{dt} = 2u - v$$

In this approximation, $u = x - 1$ and $v = y$.

SLSs of this linear system then become important tools for analyzing these linear approximations, and hence for studying the original non-linear system. Such analyses are invaluable to both the engineers who design the building and the civilians who occupy it.

Math education literature currently provides a sparse treatment of systems of ODEs, but there is a growing demand in the classroom to understand the methods and solution strategies that students invent so that teachers can continually improve their teaching methods and provide students with more meaningful activities geared toward developing their mathematical reasoning skills (Rasmussen & Keynes, 2003).

Traditional methods of instruction teach students that mathematics is a mostly static body of knowledge engineered by mathematicians and not subject to modification or

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[2] Linear systems are of the form $\begin{array}{l} \frac{dx}{dt} = ax + by \\ \frac{dy}{dt} = cx + dy \end{array}$ where *a*, *b*, *c*, *d* are real-valued constants. In this study, the focus is on SLSs that arise from these systems. Any system of differential equations that is not of the above form is called "nonlinear." Some nonlinear systems can be linearized around certain fixed points such as the "swaying skyscraper" system of ODEs. In this case, the nonlinear system can be treated as a linear system in particular neighborhoods about fixed points. In general, it is very difficult to find the general solution to a nonlinear system of ODEs.

development by students. Teachers believe that through drill and practice students can hone their computational skills which thereby increase speed and accuracy by strengthening certain stimulus-response bonds in their brains (Resnick & Ford, 1981). In this view, the teacher's primary responsibility is to provide both the appropriate amount of practice as well as the most effective ordering of the material so the student can, over time, achieve mastery of the skills that their teachers and textbook authors envisioned. Easier problems and sections are organized before harder ones and the student must climb from one level to the next as in a video game, passively absorbing the mathematics along the way. This type of mathematical learning is not only common to differential equations instruction, but pervades all levels of math ranging from elementary school to college mathematics. Determining what prerequisites (Gagné, 1965) are necessary for students to be able to learn new material is the focus of many textbook authors and teachers. According to Gagné (1965), "Control of the external events in the learning situation is what is typically meant by the word *instruction*" (p. 303). This type of external control is evident in many traditional classrooms where the teacher presents mathematics as a collection of rules to be memorized. The students are drilled frequently and then tested on their level of retention and skill mastery as demonstrated by their performance on exams. The responsibility of the curriculum materials, often under the supervision of a teacher, is to determine whether or not the student is ready for the next skill level by comparing the student's score against some established standard (Erlwanger, 1973). Instead of trying to uncover the psychology behind the student's reasoning on these problems, teachers focus on their students' mistakes made by misapplying certain algorithms.

Traditional instruction tends to discourage students from creating their own strategies and techniques for solving problems. One promising alternative to traditional curricular approaches is the realistic mathematics education approach (RME) (Freudenthal, 1991; Gravemeijer & Doorman, 1999; Rasmussen & King, 2000; Yackel, Stephan, Rasmussen, & Underwood, 2003), which is geared toward helping students invent their own methods of reasoning and solution strategies, enabling a conceptual approach to learning not based strictly on retention and memorization like its more traditional counterparts. The primary vehicles for learning in RME are the principles of guided reinvention and emergent models, two techniques that draw on students' individual and collective experiences in both abstract and realistic mathematical environments.

Freudenthal (1991) explains that in RME students learn new material by mathematizing specific situations and mathematical activity. He defines mathematizing as, "The process by which reality is trimmed to the mathematician's needs" (p. 30). He then outlines two types of mathematizing: horizontal and vertical. Horizontal mathematizing can be characterized by mathematical contexts that are embedded in real-world situations; it is accomplished by formulating problem situations so that they can later be subject to mathematical analysis. Vertical mathematizing, on the other hand, strictly reflects students' learning activities that have their foundations in horizontal mathematizing. The new mathematical realities that result from vertical mathematizing can serve as a foundation for more horizontal mathematizing and the process continues in a cyclical manner (Freudenthal, 1991; Rasmussen, Zandieh, King, & Teppo, 2005). Rasmussen et al. (2005) modified this notion of mathematizing for undergraduate mathematics in two ways. First, the initial problem situations may not be real world contexts, but mathematical contexts that are experientially real for learners. Second, mathematizing refers to forms of reasoning such as defining, algorithmatizing, and symbolizing rather than concepts like functions or long division (Rasmussen et al., 2005; Zandieh & Rasmussen, 2005).

In RME, students are provided with multiple opportunities through different learning contexts to reinvent solution strategies for certain types of problems. Their learning is grounded in experientially real situations, which ultimately lead to their development of formal mathematics. Rasmussen & King (2000) cite that identifying what is experientially real for students depends entirely on their background and experience. Nonetheless, whether the student is studying differential equations at the college level or addition and subtraction at the elementary school level, realistic mathematics education can be used to guide their intellectual development (Yackel et al., 2003).

Take, for instance, a study conducted by Smith III (2002) in which he applies a RME theoretical perspective to analyze and understand how students reason about fractions. He found that children underwent two major stages of development in their knowledge of fractions by constructing meaning for them by linking quotients to divided quantities and exploring their mathematical properties as numbers. He explains, "In conceptual development, it matters little whether we require students to learn the official terms *numerator* and *denominator*…what matters is whether students understand that the meaning of the two numerical components is given by their position” (p. 7). In order for them to understand that concept, they must mathematize their previous work with fractions and ratios. Manipulative materials, such as fraction tiles and fraction strips, allow the student to experience the mathematical environment necessary for proportional reasoning; it is through this experience that they are horizontally mathematizing (Freudenthal, 1991; Smith III, 2002). The act of partitioning these materials into divided quantities and developing meanings for fractions is an act of vertical mathematizing developed as a product of the previous horizontal mathematizing.

In designing an effective instructional sequence for building fraction understanding, the teacher must be aware of students’ perceptions of fractions as instructions for dividing and as a divided quantity. Armed with this knowledge, the teacher can then design in-class experiments geared toward uncovering what students think and know about fractions and then use the results to design activities aimed at building on their previous knowledge.

The previous example illustrates the idea of self-generated models, a principal design heuristic of RME (Gravemeijer, 1994). In these emergent models, meaning is derived cognitively through the students’ experiences with the mathematical material and not from external entities. According to Yackel et al. (2003), “The aim of instruction is to provide opportunities for the learner to develop self-generated models of a situation that can later evolve into more sophisticated models for their mathematical reasoning on a more formal level” (p. 105). A second design heuristic of RME is the principle of guided reinvention (Rasmussen & King, 2000; Yackel et al., 2003). Through an examination of students’ informal solution strategies and historical trends, teachers can design activities geared toward helping students reinvent mathematical knowledge through their experience with the mathematics. Curriculum development can be viewed as a series of activities that make sense to the designer based on these observations (Gravemeijer, 1994).

Applying these curriculum design heuristics into the differential equations classroom affords an increased opportunity to gain an understanding of student reasoning that might not be possible under more traditional models of instruction (Rasmussen, 2001). Instead of having an exchange based exclusively on whether or not the student applied the correct rule at the right time, the teacher can try to negotiate norms for argumentation (Yackel & Cobb, 1996) with the students as they increasingly become comfortable sharing their understandings of concepts instead of worrying about whether or not they have obtained the "right" answer. Through prediction, exploration, mathematization, and generalization (Rasmussen & Keynes, 2003), students construct their own meanings for SLS to systems of ODEs. The focus is on experiencing productive classroom *discussion*, moderated by the teacher. Through these classroom activities and discussions, useful data was obtained for understanding how students conceptualize solutions to systems of two linear ODEs.

CHAPTER 2

REVIEW OF THE LITERATURE

This chapter details the theoretical background for this study, focusing on RME-inspired instructional design theory, learning theories, and a review of the literature on proportional reasoning and differential equations. The chapter begins by highlighting some major motivations for conducting this type of differential equations specific research and by discussing similar mathematical studies.

Designing meaningful instructional sequences for students is at the core of many reform curriculums. RME-inspired instructional methods afford students the opportunity to develop and express their own understanding of mathematics by having them set up and solve problems that are experientially real for them (Gravemeijer, Cobb, Bowers, & Whitenack, 2000; Gravemeijer & Galen, 2003). What is considered to be experientially real depends on both the students' mathematical maturity and background (Gravemeijer & Doorman, 1999). For instance, a word problem describing the rate of change of fish in a pond over time would likely take on a very different meaning for differential equations students than it would for students in an elementary school math class. In the differential equations class the concept of "rate of change," rooted in students algebra and calculus knowledge, would likely form the experientially real concept for them whereas in the elementary school math class the focus would likely be centered around counting the number of fish in the pond since students at that level do not usually have much exposure to the idea of rate of change.

Sfard's (1998) *participation metaphor* (PM) and *acquisition metaphor* (AM) dichotomy[3] appears to shed some light on the learning process. She claims that participation in a community of learners ought to be treated as a literal construct and not merely as a

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[3] Sfard characterizes the AM by the inward movement of knowledge and the PM by the shifting bonds that form between the individual and others in a community of practice. She believes that the PM and AM operate reflexively with respect to each other.

figurative entity. According to Sfard (1998), "Indeed, PM makes salient the dialectic nature of the learning interaction: The whole and the parts affect and inform each other" (p. 6). Just as the AM focuses on the mechanisms at work in an individual's mind, the PM applies those constructs to the community of learners. The PM acknowledges that the identity of an individual is always grounded in their position within the community. Conversely, the practitioners of knowledge cannot function without ongoing contributions and communications from the individuals who make up the community (Sfard, 1998).

Linear algebra researchers have also found motivation for using an inquiry-oriented instructional design approach instead of more traditional teaching methods that emphasize rote memorization of definitions and algorithms. A major hurdle that confronts students in these traditional classes is what Dorier (2000) calls the *formalism obstacle* which is recognized by both teachers and students as a collection of conceptual and practical difficulties resulting from a lack of awareness of the unifying and generalizing nature of formal concepts. He suggests that for instructional activities to be effective they must "reconstruct an epistemologically controlled genesis, taking into account the specific constraints of the teaching context" (p. 102). In this sense, understanding the nature of mathematical knowledge is necessary to construct activities that tend toward the development of emergent models for the students.

Rogalski's (2000) teaching experiment in Lille sought to redesign the teaching of linear algebra to undergraduates. Although Rogalski did not use the instructional design heuristics of guided reinvention and emergent models, his instructional design is highly compatible with these heuristics. In designing his course, he focused on the formalizing, generalizing, and simplifying nature of the concepts of linear algebra in addition to developing an awareness of the different types of mental schemes[4] that students bring with them into the linear algebra classroom. The problem situation was trying to get students to visualize the Gaussian elimination method by encouraging the development of emergent

[4] Rogalski does not make use of the term "mental schemes" in his paper, but he does cite the importance of being aware of the prerequisite materials students bring with them into the linear algebra classroom. He refers to these 'prerequisites' as being more of a 'state of mind' or 'competency' than a specific body of knowledge. To avoid possible reader confusion over the traditional connotations "prerequisites" has, I instead use "mental schemes" to descriptively reflect the holistic nature of this prerequisite knowledge.

models that would provide a geometrical intuition of certain algebraic characteristics of linear equations. The experiment showed that critical thinking and reflective abstraction about linear algebra by students was brought about by a sequence of carefully designed teaching activities concerning the epistemological nature of linear algebra (Rogalski, 2000). This observation is consistent with the RME view that modeling and symbolizing form an integral component of the mathematical conceptualization of a problem situation (Gravemeijer & Doorman, 1999).

INSTRUCTIONAL DESIGN THEORY (RME)

A key feature of RME is having students collaboratively engage in mathematical activities so that they can reinvent mathematical knowledge for themselves. Participation in these activities forms the basis for advancing mathematical understanding through the progressive development from producing informal algorithms to producing formal mathematics (Rasmussen et al., 2005). Experientially real problems allow meaning to evolve naturally out of students' mathematical activity with familiar and realistic situations. Emergent models develop from these problems and are centered on a situation that serves as a *model of* a mathematical idea and progressively develops into a *model for* mathematical reasoning (Gravemeijer & Doorman, 1999). A key part of developing emergent models is being able to make connections with previous mathematical knowledge with familiar mathematical notions and then using them as tools to develop new ways of solving problems. According to Brousseau (1997), "A piece of knowledge is the student's adaptation to a situation...which 'justifies' this piece of knowledge by making it more or less effective, of different pieces of knowledge leading to learning and of the performance of tasks of different complexity" (p. 98). Brousseau makes an important contrast between knowledge and knowing. He suggests that knowledge is not merely a static entity to be collected and used, but is a dynamic process of adapting to different situations. The "knowledge" he discusses appears to be more of an act of "knowing" much in the same light as Rasmussen et al.'s (2005) advancing mathematical activity represents a contrast to acquisitionist accounts of "advanced" mathematical thinking. Past-tense adjectives or nouns like "knowledge" and "advanced" convey a static nature to knowledge whereas contemporary theories of learning argue for learning as an evolving, adaptive process that can be characterized more accurately by the gerunds "knowing" and "advancing" (von Glaserfeld, 1990).

Gravemeijer (1994) describes an instructional sequence where a ruler that is initially used as a "*model of* iterating measurement units" evolves into a "*model for* reasoning about mental computation strategies with numbers up to 100" (p. 118). The ruler initially serves as a tool for measurement; its manifestation dynamically transforms as students progress in their measuring activities. Students' mental schemas of measurement are acted upon through ongoing internal dialogues[5] with the ruler (Brousseau, 1997; Gravemeijer 1994; von Glaserfeld, 1995). Symbolizations on the number line thereby function as a basis for reasoning mathematically about more complicated ideas and problems involving number relations. New solution strategies are formed by students through guided experimentation and can then be used to further investigate more complicated problems (Brousseau, 1997). Constructing emergent models such as the ruler model often provides students with an invaluable sense of ownership over their ideas that can in turn contribute significantly to their mathematical confidence (Gravemeijer, 1999).

Another key feature of RME is the concept of didactising, which can be defined as the process of designing and organizing instructional activities in specific content areas with the goal of having students develop their own strategies and solution methods while simultaneously providing material for the instructional designer's toolkit (Yackel et al., 2003). Both the teacher and students continually negotiate didactical contracts (Brousseau, 1997) with each other so that they experience reciprocal relationships with each other. During the course of the academic semester, the teacher is influenced by the students' reactions to their lesson which in turn helps them formulate future lessons in a cyclical manner. In this way, the students inform the practice of teaching just as much as teaching informs the students' practice of learning. The didactical contract therefore serves to provide the teacher with freedom and creativity in their lesson planning and ensures that their

[5] Brousseau characterizes an internal dialogue as a "dialectic of action," in which the learner's strategies can be viewed as propositions accepted or rejected by experimentation in situational dialogues. In terms of emergent models, this dialectic of action serves as a *model for* relationships between certain objects that are perceived as relevant for specific situations.

curriculum development is an ongoing, dynamic process continually informed by the student body.

Horizontal didactising is the research activity of using the results of others' studies to guide and inform the construction of new instructional design materials. Vertical didactising is concerned with documenting and analyzing classroom events for the purposes of guiding future teaching methods (Yackel et. al, 2003). Both types of didactising are complimentary to each other and allow for an ongoing process of curriculum evaluation and reform. Pre-images form the basis for horizontal didactising and after-images are the results of the studies that can guide the development of future didactising efforts.

For Yackel et al.'s study, the pre-images used to guide their study were grounded in previously established RME principles. According to Yackel et al. (2003), "This paper serves as an example of horizontal didactising in that we used the after-images of other RME designers as pre-images to develop instruction in a new situation" (p. 109). They designed an instructional sequence called the "Stacking Cubes Sequence," which was created to help students develop notions of standard symbolic expressions and graphical representations simultaneously with the idea of a rate of change. The students surveyed in their study were college-age young adults who had recently graduated high school and had completed at least one year of algebra. When presented with linear situations in graphical and symbolic settings, these students were initially unable to reason about them in a meaningful way. Yackel et al. (2003) claim that these conceptual difficulties are likely the product of a traditional instructional sequence in which prescriptive measures are given to the students for how to graph linear equations in the Cartesian plane. The learner is expected to see certain mathematical relationships, but from an RME design perspective the Cartesian plane model does not implicitly contain those relationships. Instead, students experience meaningful mathematical relationships through self-generated models[6] where there is a dialectical relation between symbol use and mathematical reasoning (Gravemeijer, 1994; Yackel et al., 2003).

[6] Self-generated models refer to internalized models of learning that continuously evolve from a *model of* a situation into a *model for* reasoning.

In the Stacking Cubes sequence, students are provided with an experientially real situation concerning the construction of towers of certain heights. They are then asked to determine the height of various towers in a sequence. According to Yackel et al. (2003), "The goal is that the learner reifies his/her models of the tower sequences so that their activity with the graphs becomes objects in their own right, or models for reasoning about linear growth" (p. 105). This goal contrasts that of the traditional instructional sequence in which students are asked to use the Cartesian plane as a model for reasoning about linear growth when the Cartesian plane itself does not have any implicit meaning to them. Furthermore, students involved in the Stacking Cubes sequence were able to establish a sense of ownership over their ideas as they figured out mathematics for themselves, a sign of successful application of the guided reinvention instructional design heuristic by the instructors (Gravemeijer, 1999; Yackel et al., 2003).

To confront students' difficulties with algorithmic processes such as long division and the arithmetic of fractions, several RME-inspired researchers developed a didactic approach called progressive schematization to demonstrate how an informal alternative approach to some traditional acquisitionist methods might be better suited for school practice. The first characteristic of progressive schematization is schematic notation and chunking[7]. The second characteristic is the use and application of experientially real, concrete contexts as problem-solving environments. The idea of progressive schematization represents a modification of Treffers' (1991) broader progressive mathematization (van Putten, van den Brom-Snijders, & Beishuizen, 2005) which was subsequently modified by Rasmussen et al. (2005) under the context of advancing mathematical activity[8]

Advancing mathematical activity is a term used to describe the process of students' cumulative activity. This term expresses the dynamic nature of learning and provides a contrast to stage-based learning in which the focus is on having students obtain and utilize a static body of categorical knowledge. The word activity is used instead of thinking because

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[7] There are several different levels of chunking, ranging from low-level to high-level. What distinguish different levels of chunking are the levels of sophistication the students employ in their solution strategies.

[8] For a brief discussion of progressive mathematization, see Freudenthal's synopsis earlier in Chapter 1 of this thesis. On this same page, you can also find a summary of Rasmussen et al.'s modifications to the idea of progressive mathematization.

of the way mathematical reasoning is grounded in participation in socially and culturally situated mathematical practices (Rasmussen et al., 2005). The idea of advancing mathematical activity is highly compatible with Sfard's (1998) PM which states that learning a subject is ingrained in the process of becoming a member of a particular community. Participation in these practices and subsequent modifications of them is what constitutes learning. The depth and complexity of these practices evolves via progressive mathematization, with its horizontal and vertical layers of activity. Symbolizing, algorithmatizing, and defining are the major mathematical practices behind the evolution of ideas (Gravemeijer et al., 2000; Rasmussen et al., 2005).

For a detailed example of advancing mathematical activity, consider a study conducted by Rasmussen et al. (2005) in which students worked on several problems to determine the long-term behavior of a population for various initial conditions. On one particular problem, students were given a graph of the differential equation, but they did not have access to the equation itself. Certain mathematical constructs such as the phase line[9] emerged from student thinking. Through progressive mathematization, the meaning of these constructs changed. For example, one student, Kevin, viewed the phase line as representing the evolution of all solution functions to this particular differential equation. In a subsequent problem, another student, Joaquin, mathematized previous symbolizing activities such as this one to perform a bifurcation analysis of a differential equation with a parameter by distinguishing among five different phase line representations for solutions. Instead of using the phase line to represent the evolution of all solution functions to a particular differential equation, Joaquin use phase lines as tools for reasoning about the types of solution functions that corresponded to different parameter values. He developed a more dynamic view of the phase line than the previous student since he developed five different kinds of phase lines to suit each particular class of parameter values. Rasmussen et al. (2005) explains how advancing mathematical activity encompasses the practice of symbolizing:

> "Indicative of horizontal mathematizing, the phase line was first used as a device to record and communicate student reasoning and conclusions. In the latter part of the second example, this symbol became a tool for reasoning about the

[9] The phase line is a mathematical construct that illustrates the long-term behavior of solution functions to first-order ODEs in particular regions either above or below equilibrium solutions.

> generalized space of solution functions in a dynamic manner. Thus, in relation to students' previous activity, this shift represents a form of vertical mathematizing that builds on and extends previous horizontal mathematizing activity" (p. 63).

The symbol of the phase line evolved through the mathematical activity of the students in progressively more complex problem situations. Students symbolized the phase line both in the context of their mathematical activity as well as previous notions and representations of it.

Another important feature of advancing mathematical activity is the concept of algorithmatizing, which is the practice of examining the activities that lead to the development of artifacts and algorithms. Algorithmatizing focuses on the dynamic nature of algorithms in contrast to certain acquisitionist views of algorithms which are focused on the collection and "correct" execution of various facts and procedures. Rasmussen et al. (2005) illustrate this concept through a teaching experiment in which students were asked to mathematically model the number of fish in a pond with a growth parameter given various initial conditions. They were not supplied with any algorithms for this task. Students used their previous knowledge of rate of change to generate tables and graphs for organizing their information. According to them, "With rate of change and the context situation as the ground for horizontally mathematizing the problem to develop a procedure, the students had a basis for developing algorithms; that represents a progression or advancement of their mathematical activity" (p. 65). In order for students to develop meaningful algorithms, their thinking must be grounded in experientially real problem situations.

Rasmussen et al. (2005) claim that defining can also function as a dynamic, ongoing process. They distinguish between two major types of defining inspired by Freudenthal. Descriptive defining is the process of focusing on certain characteristic properties of an object to form a specific description of the object itself. Constructive defining, on the other hand, is focused on using familiar objects to model new ones. As an example, they asked various math and science-oriented undergraduates to define various geometric concepts including a triangle. They assert, "This type of activity builds on the previous organization to 'define' and create this new concept for themselves through the activity of examining examples" (p. 67). New definitions for familiar terms are continually revised and changed as students engage in mathematical activities across different contexts. In this sense, the definitions are both context-dependent and specific to the individual mathematical realities of the students.

A hypothetical learning trajectory (HLT) is primarily concerned with how the teacher determines what the overall learning goals of the students ought to be and how they should design tasks connected to student thinking and learning. The main goals of HLT's are to conjecture instructional activities that will bring about certain mental processes or actions that move students through a learning cycle in which they can achieve mastery of specific goals within a particular mathematical domain (Clements & Sarama, 2004).

For an example of the use of HLT's, consider a study by Clements, Wilson, & Sarama (2004) in which they examined the type of tools children develop in order to construct geometric shapes. According to them, "The ability to describe, use, and visualize the effects of composing and decomposing geometric shapes is a major conceptual field and set of competencies in the domain of geometry" (p. 164). Their focus is on developmental progressions made by students as a result of participation in instructional activities. In this way, the use of the HLT appears to bring about notions of advancing mathematical activity and Sfard's PM (Rasmussen et al., 2005; Sfard, 1998). Children were given access to both software and print materials. One of the major goals of this study was for the researchers to construct meaningful and enriching mathematical software that teachers could use in their classroom. Results of the study suggest that students given tasks involving the composition and decomposition of two-dimensional figures demonstrated developmental progressions consistent with their hypothesis that children in higher grades and with more mathematical experience would score higher on their geometry tests than the younger students. These observed developmental progressions were not only useful for further curriculum and software development, but they also provided opportunities for the development of refined HLT's (Clements, Wilson, & Sarama, 2004).

Each of the RME-inspired constructs outlined in this section supplies important theoretical grounding and motivation for our differential equations study.

LEARNING THEORIES

My theoretical orientation comes from a socio-constructivist view in which individual math activity is related to emerging classroom meanings and practices (Cobb & Yackel, 1996; Sfard, 1998). The central focus of this orientation is on students' mathematical activity, beliefs, and argumentation. Constructs from this perspective that help organize these include social and sociomathematical norms, chains of signification, and the acquisition and participation metaphor for learning. After discussing these constructs, I conclude with a summary framework that details how each of these constructs work together.

In particular relation to classroom discourse, the construction of social and sociomathematical norms highlight the nature of argumentation. Social norms are taken-as-shared beliefs that constitute a foundation for communication and allow for classroom interactions to flow smoothly. According to Yackel & Rasmussen (2002), "What becomes normative in a classroom is constrained by the current goals, beliefs, suppositions, and assumptions of the classroom participants" (p.316). These norms are formed across a variety of situations in which the students' role, their perceived role of others, and their beliefs concerning the general nature of mathematical activity evolve. For instance, during the first few days of class a teacher may continually call students up to the board to solve problems. Suppose that during this activity, the instructor conveys a positive attitude toward the students and encourages them to experiment with their own methods and to feel free to make mistakes. So long as the students agree to participate in this way, several important norms concerning their role, the role of the teacher, and the nature of mathematical activity will develop. Students will likely anticipate having to go up to the board to explain their thinking and might be less afraid to make mistakes than they were before. Activities, such as calling students up to the board regularly to explain their thinking, are what cause normative expectancies to develop and guide the students' attitudes and beliefs toward future classroom activities and discussions.

Situated within the larger context of the social norms of the classroom community, sociomathematical norms detail what is considered to be an acceptable, elegant, efficient, or different mathematical explanation and justification in the classroom as interactively constituted by the teacher and students. These norms are formed in any classroom regardless of the instructional traditions present (Gravemeijer, et al., 2000; Rasmussen, Yackel, & King, 2003; Yackel & Cobb, 1996). According to Yackel & Cobb (1996), "One of the teacher's roles in an inquiry-oriented classroom is to facilitate mathematical discussions. At the same time, the teacher acts as a participant who can legitimize certain aspects of the child's mathematical activity and implicitly sanction others" (p. 466). They detail two major types of sociomathematical norms: mathematical difference and mathematical sophistication.

Establishing an atmosphere of awareness and acceptance of the different ways in which students solve problems in the classroom is what constitutes mathematical difference. In trying to negotiate this norm, the teacher must encourage development of an inquiry-oriented atmosphere so that students can gain a better understanding of what constitutes a mathematically different solution. One way to negotiate mathematical difference in the classroom is to have students engage in group discussion about how to approach this problem and then reconvene as a class to discuss their ideas. The teacher could then call on particular students to share their ideas with the class. In developing taken-as-shared meanings of what is mathematically significant in this problem situation, the teacher must publicly discriminate among different explanations. The teacher should be able to distinguish between students' re-voicing of previous explanations and the construction of new and novel methods. In formulating their own explanations, students must also be able to identify similarities and differences between their reasoning and the ideas of others. For instance, a beginning algebra college teacher might ask his students how to find the equation of the line passing through two points. Several students might respond with similar explanations about finding the slope of the two points and then using the point-slope form of the line. Other students might find the slope and then reinvent the slope-intercept form of a line and use that instead. The teacher must be able to negotiate meanings with the students from these explanations and help them identify the similarities and differences among them while at the same time encouraging the development of new, perhaps more efficient solution methods, some of which build off of previously established ones. All of these ideas are then put into perspective by both the teacher and students. Once the explanations and justifications for the problem have been presented and analyzed, the teacher and students can interactively decide what information is mathematically sophisticated.

The norm of mathematical sophistication is less explicit in nature than mathematical difference because teachers usually do not place a marker on what they consider to be the "best" explanation since doing so would go against the guiding principles of these norms. It is much easier to simply identify different solution methods than it is to decide which ones promote the most efficient and effective mathematical understandings. The teacher and students must collectively agree on what counts as a sophisticated solution method and what promotes the most effective understanding for that particular classroom. According to Yackel & Cobb (1996), “The teacher necessarily reflects the discipline of mathematics in the classroom. Consequently, the teacher’s reaction to a child’s solution can be interpreted as an implicit indicator of how it is valued mathematically” (p. 464). If the teacher reacts positively to a student’s remark, then the rest of the class is likely to be receptive to it as well. Yackel & Cobb (1996) provide an example of a teaching episode in which the task given to students was for them to figure out the number of chips that were in a frame with a certain configuration. The teacher’s enthusiastic response to students’ explanations and justifications ultimately led to the advancement of individual student thinking and mathematical discourse. Students were able to mathematically progress from their rudimentary ideas about counting by ones toward more sophisticated concepts of ten, two-digit number partitioning, and justification for their methods.

A chain of signification is a term used to describe the various forms that a model might take by examining evolving “taken-as-shared” meanings associated with the use of tools and symbols (Yackel et al., 2003). These taken-as-shared meanings are often formed during the negotiation between the teacher and students of interpretations and solutions for experientially real problems. According to Gravemeijer et al. (2000), “The basic component of a chain of signification is a sign, where a sign is thought of as a semantic relation between a signifier and a signified. A new link in the chain is established when a sign itself becomes the signified for a new signifier” (p. 262). A signifier can be thought of as a tool, either concrete or abstract, that can be used as a referent to an object or collection of objects, called the signified. The signifier and signified function collectively as a sign. For instance, a child learning how to count might use their fingers as a signifier to signify the items they are counting. These entities function together in that eventually the child may be able to use numerals as symbols for the earlier process of counting by hand. The use of such numerals would suggest that the child was using their fingers and the items together as a sign represented by the internal structure of the numerals (Yackel et al., 2003).

Each step in the chain of signification appears to be the product of an ongoing interaction between the AM and PM. Take, for instance, the structuring numbers sequence (Gravemeijer et al., 2000) in which an arithmetic rack sequence was created to develop quantitative reasoning skills with numbers up to 20. The arithmetic rack was introduced following the researcher giving the students an experientially real problem involving a double-decker bus. Part of the instructional design plans were for students to eventually use the beads on the top and bottom rods to illustrate the number of passengers on the top or bottom decks of the bus, and have them use the moving beads to represent the number of passengers boarding or leaving the bus. There were several classroom math practices that evolved during the study, which developed hierarchically as follows: making configurations on the arithmetic rack; describing configurations in terms of fives, tens, and doubles; reasoning in terms of groups while solving addition and subtraction tasks; reasoning in terms of number relations to make and evaluate configurations; and finally, reasoning numerically to solve addition and subtraction tasks. Each classroom math practice evolved as a result of signifying the previous one. For instance, in order to develop the practice of reasoning in terms of number relations to make and evaluate configurations, students had to anticipate bead configurations in ways that would lead to an enumeration of the collection. The AM present in this transfer appears to be the students' internalized ideas of grouping and the PM appears to be the different ways they moved beads around on the rack to collectively reason toward what Gravemeijer et al. (2000) call an "anticipatory going-through-ten strategy" (p. 256). Data analysis in this study seems to suggest that the AM and PM operate reflexively at each step in the chain of signification for the double-decker bus problem.

Figure 1 illustrates the socio-constructivist learning cycle detailed in the preceding paragraphs. Mathematical models evolve and change in their meaning and presentation through the interaction of individual activity and conceptions. Through the *chain of signification,* these models can increase in both their levels of sophistication and abstraction and can be used by both the teacher and students to inform and guide future activities. In addition to guiding students' conceptual development, these models enable them to progressively formalize mathematics, establish a sense of ownership over their ideas, and engage in meaningful argumentation and classroom discourse.

A thorough probe into the interaction between classroom norms and practices and individual conceptions and activities in this study provides a means for understanding the psychological processes and motivations that underlie students' mathematical beliefs and justifications. The learning cycle outlined in Figure 1 also provides a theoretical lens for the coding scheme[10] used in the data analysis.

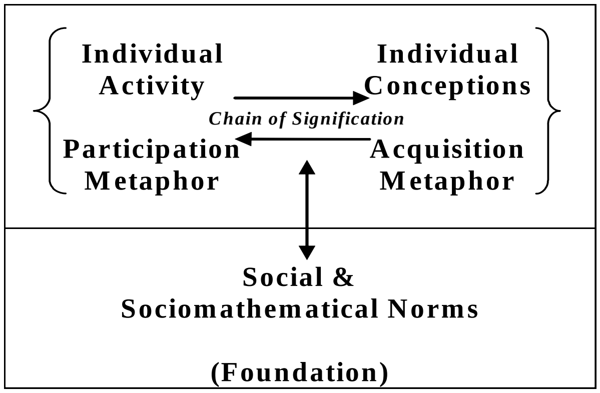

Figure 1. Socio-constructivist learning cycle.

CONTENT-SPECIFIC LITERATURE REVIEW

Proportional Reasoning

Understanding students' methods of proportional reasoning is a central goal of this research. In order to assess students' proportional reasoning abilities, teachers must prepare classroom activities that provide opportunities for them to both correctly and incorrectly apply additive and multiplicative reasoning. Different assessment methods reveal different aspects of students' thinking so it is up to teachers to design those activities that they feel will provide the aspects of proportional reasoning of interest to them (Bright, Joyner, & Wallis, 2003). According to Cai & Sun (2002), "Proportional reasoning involves 'a sense of covariation, multiple comparisons, and the ability to mentally store and process several pieces of information'" (p. 195). Proportional reasoning can be represented mathematically by direct proportionality and inverse proportionality. For instance, the mathematical equation $y = mx$ indicates a direct relationship between y and x and the equation $y = \frac{m}{x}$ implies an inverse relationship between y and x. Conceptualizing quantitative relationships

[10] For specific information on the coding scheme used in the data analysis, see Chapter 3.

and comparing ratios are common manifestations of proportional reasoning (Cai & Sun, 2002).

While multiplicative reasoning involves the use of ratios and proportions to make comparisons, additive reasoning can be characterized by comparing two quantities based on counting schemes such as sums or differences of numbers (Bright, Joyner, & Wallis, 2003). For instance, consider two pitchers, both containing orange juice. Suppose the first pitcher is made by mixing one can of orange concentrate with 3 cans of water and suppose the second pitcher is made by mixing 2 cans of orange concentrate with 6 cans of water. Somebody reasoning additively would likely think that the juice in the second pitcher would have more of an “orangey” flavor to it than the first pitcher because of the fact that the second pitcher contains one more can of concentrate and three more cans of water than the first one. Just by the sheer virtue of the second container containing a quantity that is greater in value than the first one, someone reasoning additively might think this way. In contrast, a person reasoning multiplicatively would immediately assert the equality of “orangeyness” of the two pitchers based on a proportion between the two. They would see that, proportionally speaking, there is no difference between the quantities of orange juice in the two pitchers.

Sometimes additive and multiplicative relationships can work together. In the Chinese elementary school mathematics curriculum, the concept of ratio is introduced following division with fractions. The rationale for introducing it in this way is so that students can use division as a bridge to connect the concept of ratio and how it is represented (Cai & Sun, 2002). For example, students might be asked to figure out a problem where they are given information about the number of workers in a given company of a particular gender and then are asked to figure out how that number is related to the number of workers of the other gender. Suppose that they are told that twenty men work for this company of eighty workers. In order to conclude that there are three times as many women as men at this company, the students would have to first utilize their knowledge of subtraction ($80 - 20 = 60$) and then use division ($60 \div 20 = 3$) to conclude that among these 80 workers, for every three female workers there is one male worker. As shown in the above problem, additive reasoning is employed which is then followed up by multiplicative reasoning from which multiplicative relationships are used to compare the two quantities, which establishes a ratio between men and women.

The concept of ratio serves as a foundation for understanding the concept of proportion (Cai & Sun, 2002; Lamon, 1994). At first sight, a proportion problem may appear to not be much different than a ratio problem, but as Cai & Sun (2002) explain, "The initial ratio is a ratio of two concrete quantities. By simplifying the ratio, it becomes an abstraction. This process from concrete to abstraction reflects the relationship between these two ratios" (p. 199). In the Chinese mathematics curriculum, proportions are defined as two ratios that have the same value. They use *scale drawings* which are intended to depict real world scenarios such as a car traveling over a certain distance in a given amount of time. Emphasizing the role and meaning of scale in measurement of distance over time appears to be an effective way for helping students to transition from ratio to proportion. For example, suppose students are given that Car #1 travels 100 miles in 2 hours and Car #2 travels 250 miles in 5 hours and they are asked to determine which car traveled faster. Using their previous knowledge about ratio, they would divide 100 by 2 and 250 by 5 to ascertain that both cars traveled at the same speed. Through a scale drawing they would be able visualize these equal ratios even though one has numbers that are larger than the other, an idea that often prevents students from understanding proportions (Cai & Sun, 2002; Lobato & Thanheiser, 2002).

Lamon's (1994) concept of unitizing appears to help shed light on some of the psychological processes at work behind students applying their knowledge of ratio to developing an understanding of proportionality. According to Lamon (2002), unitizing is "the ability to construct a reference unit or a unit whole, and then to reinterpret a situation in terms of that unit…[which] involves the progressive composition of units to form increasingly complex structures" (p. 92). Unitized mathematical strategies and techniques illustrate levels of progressive chunking which appear to be highly compatible with those found in the chain of signification. For instance, consider a child's first encounters with the activity of counting. Children often count on their fingers in early childhood and gradually learn how to count along out loud. Eventually these words come to represent the quantity that before was just a visual symbol. As students gain more experience with counting and counting activities, they form more sophisticated strategies such as counting by fives, counting by tens, and so on. These increasingly sophisticated strategies are likely the result of progressive mathematization of previous counting schemas and indicate higher levels of conceptual organization. Often students engage in norming of their strategies, a method of reconceptualizing with respect to some previously established standard (Lamon, 2002). For instance, a student asked to imagine that California is the size of small piece of paper would have to reconceptualize the United States in terms of that definition. Other states would appear as larger or smaller pieces of paper scaled to the appropriate dimensions.

Similarly, students might encounter both unitizing and norming of their mathematical problem-solving strategies involving ratio and proportion. Using sophisticated multiplicative relationships to make sense out of a problem involving ratios such as the one involving the ratio of women to men at a certain company involves unitizing of previous additive and multiplicative reasoning schemes. Likewise, the use of percentages requires students to engage in norming fractions via a proportion to a fraction that has 100 in the denominator. Students also have to norm when confronting environments including those presented in computer software packages that scale down representations of real-life scenarios.

To assess students' understanding of the concepts of ratio and proportion, Lobato & Thanheiser (2002) designed two teaching experiments involving nine high school students over the summer term. Geometer's Sketchpad (GSP) and Mathworlds were the two software environments in which they had students operate and deal with questions concerning measuring slope. Instead of being given an explicit formula for the slope of a line, students were given real-world problems to work on. This task bears some resemblance to the scale drawings used in the Chinese elementary mathematics curriculum. One task given to the students was to draw two non-identical hills with the same steepness. Some students did not think this was possible since they thought having a steeper hill meant it would take longer to climb and since it was higher, there was no way that the two hills could possibly have the same slope. Through further interactions with the computer software these students were able to alter their notions about steepness, including coming to an agreement that the length or height of the ramp alone was not a valid measure of steepness, but it was necessary to examine the ratio of these two quantities with respect to each other.

The use of computer environments including those offered by GSP and Mathworlds helped students understand the ratio as a measure instead of as two unrelated quantities. For instance, students who previously did not understand what meaning there was in saying that someone traveled 10 centimeters in 4 seconds were now able to understand the implicit ratio of 2.5 centimeters per second contained in that statement. According to Lobato & Thanheiser (2002), “Computer environments can play an important role in the construction of a ratio by helping students conceive of a situation dynamically” (p. 171). Through experimentation with animated characters in GSP, students could guess, check, and identify patterns that emerged from the construction of different scenarios. For example, they were able to observe that 10 centimeters in 4 seconds represented the same speed as 30 centimeters in 12 seconds during a GSP activity constructed by the teachers. Further reflection and discussion of these ideas helped solidify the idea of ratio as a measure in students’ minds (Lobato and Thanheiser, 2002).

It is important to note that computer environments such as the ones used in the previous study cannot function effectively in the learning process if they are viewed as a means to an end. Rather, teachers ought to design interactive classroom activities involving the use of computer programs that encourage students to develop sophisticated arguments and justifications about ratio and proportionality. Developing this kind of classroom community in which there is a fluid interaction among the students, teacher, and software leads to a more dynamic approach to teaching and learning and results in the ongoing development of a language and theoretical orientation that can explain both how and why certain manipulations on slope had the effects they did. Formal mathematics concerning ratio can then evolve from these interactions within the classroom community (Bowers, Nickerson, & Kenehan, 2002; Lobato & Thanheiser, 2002).

Differential Equations

Currently there exist few published studies related to systems of differential equations; however, there is a growing body of literature indicating the developing need to account for student thinking and learning in systems of differential equations (Trigueros, 2000, 2001; Whitehead & Rasmussen, 2003). The theory of differential equations has evolved algebraically, numerically, and graphically throughout the past couple of centuries with the algebraic approach dominating most of the time. Recent advances in dynamical systems and the computational and graphical power of computers, however, have greatly changed the nature of differential equations over the past twenty years. Artigue (1992) argues that teaching exclusively from the algebraic approach convinces students that there is a collection of specific recipes available for solving differential equations and that the creation of these recipes is the primary goal of researchers. According to her, “The competencies required to associate given drawings and equations are not the same as those required to interpret given drawings, to predict the phase portrait of an equation, or to justify conjectures” (p. 113). Recent research in differential equations aims to shed light on the nature of these competencies. In the paragraphs below I briefly review this research and reflect on how it informs and relates to my own study.

As part of a semester long classroom teaching experiment, Whitehead & Rasmussen (2003) conducted semi-structured individual interviews with the purpose of characterizing students’ reasoning about differential equations from a dynamical systems perspective. Similar to the socio-constructivist interpretative framework used in my study, their analysis highlights individual activities and conceptions within a more encompassing perspective that coordinates psychological and sociological perspectives. The problems, which were novel for the students, were given to them after several weeks of instruction on single, first order differential equations but prior to any instruction on systems of differential equations. Three themes were developed to identify students’ mathematical conceptions and understandings of differential equations: rate use, quantification as a mental operation, and function-variable scheme. They found that students use rate as a tool to construct other mathematical models, as a ratio of displacement and time, and to make predictions about how functions change over time. Students quantified by performing mental operations on existing quantities in order to construct a new quantity, in this case the rate of change of populations from the population quantities themselves. They also developed relatively sophisticated understanding of mathematical objects as both functions and variables simultaneously, evident in explanations concerning the population values as both quantities and functions depending on the context in which they were being interpreted. In addition to this theoretical analysis being of practical value to both teachers and researchers, it also offers important insights into student thinking and justification about systems differential equations prior to formal instruction.

In another study on student learning of systems of differential equations, Trigueros (2001) analyzed the strategies that students use when working in a graphical context. She collected data from two differential equations classes at a private university. One class consisted of 34 Applied Mathematics students and the other consisted of 37 Economics students, both taught by her. Individual task-based student interviews were performed throughout the semester focusing only on material covered in class. Data analysis focused on interpreting student strategies; in particular those that helped students make sense of and reason about the given tasks, solve problems, and interpret solutions. She found that students had a tendency to ignore the dynamics of systems of ODEs and provided a static description of the behavior of solutions in the phase plane causing them to experience difficulties relating graphs of the same solution curve in different geometrical interpretations. When not strictly working with the analytical properties of systems, students also failed to make sense of their geometric properties.

While my study is inspired by the work of Trigueros (2001) and Whitehead & Rasmussen (2003), my research is focused on innovative classroom interaction in which students successfully made progress on reinventing SLSs. I am not primarily concerned with identifying the conceptual difficulties students face, but am more interested in understanding how students mathematize SLSs to systems of linear ODEs. As with Whitehead & Rasmussen's (2003) study, my research is also focused on identifying and understanding the themes that emerge from the dynamic interactions of individual student activity and conceptions situated within the context of the classroom community.

Several key studies on students' representation and understanding of SLSs to systems of ODEs have emerged recently (Rasmussen & Keynes, 2003; Trigueros, 2004). One of the major research goals of Trigueros' (2004) study was to analyze student responses to questions pertaining to their understanding of the meaning and representation of SLSs. She collected data from a class in applied dynamical systems consisting of 38 Economics students at a small private university. This class emphasized both the analytical and geometrical representations of solutions of linear and non-linear systems in addition to the theoretical nature of these solutions. A pre-interview questionnaire was distributed to students followed by a semi-structured interview. As with her previous studies, Trigueros (2004) was focused on identifying student difficulties with interpreting and analyzing SLSs given in different forms and contexts. She found that students experienced difficulty in understanding the geometric representation of SLSs, the meaning of the characteristics of the analytical representation of the SLSs, and constructing other solutions from SLSs.

According to her, “These results…show that emphasis in relationships between concepts is not enough. More effort is needed in collaborative work between students and in the design of specific activities in which students work on making those links between concepts” (p. 133).

Rasmussen and Keynes (2003) study provides an alternative approach to finding SLSs that is more consistent with students’ conceptual development. This approach is called the “eigenvector first approach” and is a contrast to methods of instruction in which the computation of eigenvalues precedes the computation of eigenvectors. The “eigenvector first approach” comes about following an instructional sequence intended for students to develop their own method for locating lines of eigenvectors and corresponding solutions to systems of two first-order linear ODEs. One of the researchers’ major goals in this study was to identify mathematizing schemes present in student thinking and reasoning including the activities of prediction, exploration, generalizating, and algorithmatizing. Students were asked to predict graphical representations in the position-velocity plane for the motion of a mass attached to a spring. Following this activity, they used a computer program to graphically explore and track changes in the vector field as the friction coefficient was varied. Students then used those results to invent an algebraic approach for determining the slope of the SLSs followed by a generalization of their results via the determination of a closed form for the general solution for this situation. Rasmussen and Keynes (2003) found that in addition to this instructional sequence providing students with a more natural and accessible way of understanding the development and meaning of systems of linear ODEs and their solutions, students also developed a sense of ownership over their ideas.

My research goals closely follow those of Rasmussen and Keynes (2003) in that I also develop an “eigenvector first approach,” however, I am also focused on understanding how students develop and utilize the ideas of proportional reasoning in their construction of SLSs, an idea that has not yet been fully explored in the research literature. I will also try to identify how students mathematize the development of SLSs in graphical, algebraic, and numerical settings and make recommendations for the design of meaningful instructional activities.

CHAPTER 3

METHODOLOGY

The focus of this chapter is on the methods of data collection and data analysis carried out in this study. Attention is paid to both the theoretical and practical aspects of the methodology, including a descriptive overview of the techniques and theories that guided the classroom research. The chapter is broken down into four sections, the first of which is concerned with the theoretical motivations behind the design of the classroom teaching experiment. The second section provides detailed information about the classroom population and environment, including general profiles of the teachers, students, and researchers involved in this study. The third section outlines the data sources, including classroom videotaping sessions and group work while placing these sources in the greater context of the classroom teaching experiment. Finally, the chapter closes with a discussion of the data analysis procedures, including a brief overview of grounded theory and practical aspects of the socio-constructivist learning cycle relevant to the data analysis.

DESIGN OF THE INVESTIGATION

Although the nature and design of classroom teaching experiments varies, the constructivist teaching experiment (CTE) best characterizes the one conducted in this study (Cobb, 2000). This type of experiment is characterized by the teacher acting as researcher where teacher-student interactions are usually centered on small groups or on individual students. The emergent perspective (Cobb & Yackel, 1996), which describes learning as both a process of active individual construction and enculturation into a community of mathematical practice, provides the theoretical backing for the CTE. According to Cobb (2000), "Students are considered to contribute to the evolving classroom practices as they reorganize their individual mathematical activities. Conversely, the ways in which they make these reorganizations are constrained by their participation in the evolving classroom practices" (p. 310). The relationship between the mathematical activities of students and the local social world of the classroom is reflexive; that is, the ongoing interaction between the students and their environment determines the nature of mathematical activity within the classroom. Therefore the role of the student in the CTE is to continually reorganize their mathematical beliefs and assumptions through ongoing participation in mathematical activities and discourse. This dynamic view of learning is supported by Steffe & Thompson's (2000) contention that mathematics is a "living subject" and not a pre-determined body of knowledge to be imported by learners. Under this belief, students experience different conceptual realities which exist in their individual minds, but are influenced and guided by their interactions within the social world of the classroom. One of the primary aims of the CTE is to have students document and discuss these realities so that they can be analyzed later by the teacher-researcher who may use the data to revise instructional materials and guide future research.

The role of the teacher in the CTE is neither that of a passive observer nor an authoritarian figure. They do not carry a static formula for teaching, as their instructional practices are continually revised and reorganized based on their interactions with students during class periods. The classroom is viewed as the primary learning environment for teachers and researchers, and so the focus of a CTE could be on teacher learning in the social context of the classroom.

Another important issue pertaining to the CTE is the development of instructional sequences and activities. According to Cobb (2000), "An analysis of students' learning in a social context requires that the instructional activities be documented as they are realized in interaction" (p. 313). Therefore an essential component of the CTE is frequent de-briefing sessions in which the teacher meets with his research team and discusses what revisions ought to be made to the current instructional materials as well as what kind of activities would fit well into the next set of materials for the following day. This cyclic process of design and analysis continues with each CTE until there is a well-defined development of a local instructional theory that provides the foundation for an instructional sequence (Cobb, 2000).

The next section describes the nature and design of the classroom environment depicted in this study. Details will be provided about classroom demographics and the social norms of the classroom including the roles of the teacher, students, and researchers and how they fit into the overall scheme of the CTE.

CLASSROOM POPULATION AND TEACHER PROFILE

This study was part of an ongoing eight-week differential equations research project conducted at a large, public university in Southern California. Data collection was carried out in an elementary differential equations class taught in the Spring Semester of 2005 by a full-time professor, Chadley, of math education. A primary goal of the course was for students to develop a deep conceptual understanding of the fundamental concepts and methods for analyzing differential equations. A second goal was for students to advance in mathematical sophistication through mathematical investigation and argumentation, through the creation of personally meaningful solutions to problems, and by expanding their ways of communicating mathematical thinking and activity to others, both verbally and in writing. This course focused on the formulation of differential equations, methods for analyzing them, and the interpretation of their solutions from a graphical, analytical, and numerical perspective.

Each class session was fifty minutes in duration and the class met three times a week. The majority of the thirty-seven students were science and engineering majors with a couple of math majors. Most of the students were in their third or fourth year of school and had a background consisting of two or three semesters of calculus. Some students were concurrently enrolled in linear algebra. When asked early on in the semester about their willingness to participate in an inquiry-oriented classroom, most of the students expressed significant interest in that kind of learning environment while only a few expressed some trepidation about it. It seemed like the idea of the teacher acting as a moderator of discussion instead of as a strict authority figure was a foreign, but welcome transition for most of them. This was evident by the way they explained how in other math classes they usually did not understand the concepts behind what they were doing and that what they did learned usually lacked meaningful connections with their real world experience. Most students enrolled in this course because it was required for their science or engineering program, but many explained that they were hoping to obtain a better understanding of some of the mathematical principles they learned in earlier physics and biology classes.

One common characteristic of the teaching materials used in this course was their continuous revision and refinement. It was this quality that enabled them to be both adaptable and accessible to the students. Although there was no formal textbook requirement for this course, students were required to purchase a course reader. This reader consisted of a collection of progressive mathematical activities and RME-style problems intended to help develop students' natural conceptual development of the important ideas in differential equations. Both the organization and nature of the materials varies year to year and in some instances, week to week, based on the teacher's assessment of students' progress and conceptual understanding. There were instances in which the activities intended for class discussion on a particular day were modified following the previous class session and a new worksheet was created to be inserted into the class reader. The reasons for the modifications were multifaceted and typically a reflection of whether or not the teacher felt that they would better help advance mathematical activity on the following day.

Manipulatives and hands-on activities played a significant role in student learning and exploration during the semester. For instance, during a lesson on the shape of three-dimensional graphs of solutions to ODEs, the teacher employed pipe cleaners and red cubes to simulate the motion of a paint mark on a propeller blade of a plane moving down a runway. The students worked in groups to figure out what the shape of certain three-dimensional curves would look like by twisting and turning the pipe cleaner in various directions and by observing it from different orientations. The result was that students developed better three-dimensional visualization skills which helped them later on with describing the behavior of the SLSs in three-dimensions.

A series of java applets were also used throughout the semester to aid students in their conceptual understanding of the graphs of solution functions. Early on in the semester, students re-invented Euler's method for approximating solutions to first-order ODEs. They were able to do this with the aid of a java applet that plotted the vector field for a first-order linear ODE. By observing how the individual vectors changed with variations in time and the size of the graph, students were able to begin to distinguish between autonomous and non-autonomous ODEs. This technology also helped them to re-invent Euler's method for approximating solutions to ODEs. Later on, they used a three-dimensional java applet to aid them in their visualization of the behavior of solutions in space to systems of first-order linear ODEs. The use of this applet nicely complemented the pipe cleaner and red cube manipulative exercises.

Sometimes these java applets were displayed during class on an overhead projector screen hooked up to a SMART computer box in the front of the room. This box also enabled students to showcase their written work on a document camera that scanned their work and projected it onto the viewing screen. This technology was more convenient than traditional overhead projectors because it did not require any special paper or transparencies for use. Nearly every class session, students shared their written work with the rest of the class via the SMART box.

The classroom data used in this study came from five consecutive days of teaching that occurred on April 8, April 11, April 13, April 15, and April 18, 2005. On two of these days, April 11 and April 13, I acted as the primary teacher-researcher, while on the other three days Chadley taught the class. He had a significant teaching and research background in mathematics education, including nearly five years experience working on the revision and implementation of differential equations-specific instructional materials. This was my first in-depth study into the teaching and learning of differential equations, but I expressed interest in this type of project early on in my graduate career. I had the opportunity to teach three class sessions throughout the semester, two of which were conducted during the CTE. There were three other researchers involved in separate, but related projects. Two graduate students, Bartholomew and Opie, researched how students justified the shapes of graphs in the phase plane. Another researcher, Janis, was involved in a gesture study. Each one of them including myself was responsible for either videotaping class sessions or taking field notes during class.

Early on in the semester, Chadley negotiated productive social norms with the classroom that enabled the CTE to flourish. During the first few days of class, he had students participate in group discussion and share their ideas as a class. It was during these sessions that the norms of mathematical justification and discourse were developed. Chadley also had the students read and respond to a paper about the social culture of the classroom[11] which explained the collaborative nature of mathematics as well as the value of students coming up with and sharing their own methods for problem-solving. A typical class session

[11] See Hiebert, J., Carpenter, T.P., Fennema, E., Fuson, K.C., Wearne, D., Murray, H., Olivier, A., and Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding* (pp. 43-52). Portsmouth, NH: Heinemann.

began with a brief overview of the mathematics developed during the previous day and an introduction to a new problem or a continuation of the activities from the previous day. The rest of the class was divided up into group work sessions and classroom discussion activities. During group work, students got into small groups of three to four people and discussed problem-solving ideas with each other. After ten to twenty minutes, the teacher presided over class discussion, in which the groups shared their ideas with each other. Sometimes consensus among the class was achieved on a particular idea and other times multiple possibilities were posed from which the groups would enter further discussion to try to resolve issues brought up during whole-class discussion. At the end of class, the teacher summarized the results of class discussion and prepared students for the next day's activities.

The net effect of these collaborative teaching methods and classroom social norms was consistent student motivation toward solving novel problems and increased confidence in their problem-solving abilities. The teacher never acted as a mere authority figure, but instead as a moderator and facilitator of discussion, which enabled students to feel comfortable and to be productive in this learning environment.

DATA SOURCES

There were various methods and sources used for data collection purposes that spanned both classroom and non-classroom environments. During class sessions, the primary methods of collecting data included videotaping, audiotaping, collecting class notes, and taking field notes. The specific type of classroom data analysis that was used in this study has its roots in the theory of interaction analysis, which is based on the premise that the naturally occurring social interactions that happen in communities of practice provide the optimal foundation for analytic knowledge of the world. According to Jordan & Henderson (1995), "The goal of interaction analysis is to identify regularities in the ways in which participants utilize the resources of the complex social and material world of actors and objects within which they operate" (p. 41). By videotaping classroom events, researchers can directly review the ways in which people in groups construct knowledge and simultaneously become aware of the process of construction. Video de-briefing sessions following each class session are strongly suggested as part of the interaction analysis to try to identify patterns of reasoning and learning among the students as well as to discuss the overall effectiveness of the lesson plan for that day (Jordan & Henderson, 1995).

There were two video-cameras set up in the classroom for this study. One was positioned in the front of the class and the other was located in the back of the class. During group discussion, the cameras were focused on two groups in particular, one in the front and one in the back. The members of the front group were Efrain, Carly, and Mario and the members of the back group were Timothy, Larry, and Li. The students' desks were set up in a triangle with two students directly facing each other perpendicular to the camera and another student positioned in between them directly facing the camera. This setup allowed for the optimal viewing area of the group while still being intimate enough to capture individual expressions and gestures. Audiotapes were placed on each of the tables as backups in case the cameras experienced technical difficulties. Following each class session, students' notes were taken and photocopied for further analysis.

During whole-class discussion, the front cameraman focused on the activities and statements of the class while the back cameraman honed in on the teacher and students presenting material at the front of the class. Each cameraman made an effort not to interact with the students in any way except to ask for clarification or more information about an idea. The reason for this lack of intervention was to prevent tainting the students' data with the researchers' ideas and beliefs. There was usually someone present to take field notes during each class session to summarize the classroom events and to listen carefully to group discussion. Similarly, each student was responsible for taking notes for the duration of one class session and then doing a write-up of what they believed were the main points and purposes of the days' activities. Daily debriefing sessions were held following each class in Chadley's office for the duration of the study. During these sessions, videotapes were reviewed and the instructional materials for the next class session were usually modified in some way to help move the mathematical agenda forward. Nearly everybody in the research group participated in these meetings which offered multiple views on what seemed to work and what could be improved for the next class meeting.

Students were given weekly homework assignments that were photocopied for data collection. Several of these assignments had a discussion board component to them. Early in the CTE, students signed up for the discussion board using their last names so they could be easily identified and participate in meaningful discussion. There were seven prompts given out throughout the semester with questions geared toward probing their conceptual understanding regarding certain topics such as slope and proportionality, Euler's method, and the long-term behavior of solutions. Students were asked to reflect on class activities and mathematical concepts by engaging in meaningful written discourse. These prompts provided a useful way for the teacher to reflect on students' conceptual development and offered a great lead-in to the next class session. They were also useful for data analysis as written records that could be analyzed in the context of mathematical activities spanning several class periods. Students who normally might not speak up in class due to shyness or other anxieties were free to express themselves in this online format. In this way, the discussion board provided a meaningful dimension that nicely complemented the classroom environment.

In addition to the homework, two exams and a final were given out during the semester. Exams consisted of questions that probed students' problem solving abilities and depth of conceptual development. There were several questions that asked students to define a term in their own words or to mathematically model a physical situation via the use of appropriate equations and graphs. Very seldom did the instructor prescribe a specific method for the students to use to solve problems. For instance, when solving problems involving systems of linear ODEs on the final exam, students had the option to approach the problem with either the eigenvector-first or eigenvalue-first method so long as they justified their work. In this study, the exam data was used primarily to identify patterns and regularities in both the correctness and frequency of particular types of responses to questions.

One of the most useful methods of data analysis conducted outside the classroom is clinical interviews. One-on-one clinical interviews support an optimal environment for allowing teachers to gain insight into how students think about mathematics. In particular, they can allow for exploration into how students use and take ownership of certain ideas developed during previous classes. Because these interviews involve students thinking out loud and reasoning about both familiar and novel problems, videotapes and audiotapes of the sessions can also provide opportunities for comparison with data obtained during earlier classroom video sessions. Analytical frameworks can then be developed by researchers to account for both shifts and consistencies in student thinking across different instructional mediums and problem-solving situations (Ambrose, Nicol, Crespo, Jacobs, Moyer, & Haydar, 2004).

The clinical interviews conducted in this study were done at the end of the semester following final exam week. Twenty-one students participated in one to two hour long interview sessions conducted by one of three researchers. Each of these sessions was both videotaped and audiotaped. During the session, students were asked to read and respond to two main questions involving systems of linear ODEs. The first task involved them solving a familiar system using both the eigenvector-first and eigenvalue-first methods. Within this task was a well-defined interview protocol for the researcher to follow. What question was asked next directly depended on the students' response to the previous question. These protocols were consistent and uniform among the three researchers and twenty-one students. The second task presented a novel task for students and was also accompanied with a clear protocol. The researchers were told not to intervene with student thinking in any way except to request that they think out loud or answer questions more clearly. Of the twenty-one students interviewed, five were analyzed for this study. Two of the students were from the front group, two were from the back group, and one was a regular active participant in class, though she was not a member of either group.

Retrospective analysis and model building are essential components of the CTE methodology. According to Steffe & Thompson (2000), "Through the use of core concepts of our research program, we make concrete claims about the mental functioning of students, and these claims draw their operational character from the framework itself" (p. 297). Video de-briefing sessions are just one of several relevant examples of retrospective analysis and model building in the way that researchers collectively reflect on the events of classroom videos to form conjectures about the ways students think. Although radical constructivism asserts that mathematical realities are only accessible to the individual experiencing them, Steffe and Thompson (2000) believed that the circularity of model building between the researchers' theoretical framework and the models themselves was responsible for the continual refinement and development of new theories. The conceptual development of the researchers in a way mirrors that of the students, but is in no way mutually exclusive from theirs. Just as the students encounter perturbations in their learning and engage in reflective abstraction through their work with novel problems, the researchers analyzing the data experience the same psychological phenomenon. Reviewing past records such as classroom video, homework assignments, and exams enable researchers to continually refine their explanatory framework and construct their own theoretical realities. In this way, perhaps there is an analogue to horizontal and vertical mathematization that occurs within a community of researchers that enables them to continually refine their learning theories as they analyze classroom data. If this were the case, it would seem plausible that there would be a bridge connecting the theoritization of math education researchers to the mathematization of the students being analyzed in their problem-solving environments.

DATA ANALYSIS PROCEDURES

The data analysis schemes used in this study are deeply rooted in grounded theory (Glaser & Strauss, 1967). Traditionally, researchers form their hypotheses ad hoc and then use statistical methods to either verify or reject their initial conjectures. This method of reasoning might be well suited to certain types of physical, psychological, and mathematical studies, but the nature and design of this study required a more elastic and flexible theory that did not require the generation of hypotheses prior to data collection.

In contrast to a priori statistical methods, researchers using grounded theory form their hypotheses post hoc by trying to identify patterns and regularities in the data and then conjecturing about the nature and character of these ideas. According to Glaser & Strauss (1967), “In discovering theory, one generates conceptual categories or their properties from evidence, then the evidence from which the category emerged is used to illustrate the concept” (p. 23). The more avenues one has for data collection, the richer the data analysis will be. As researchers sift through video, audio, and written data, they create categories based on certain patterns and regularities observed in the data. From these categories emerge initial hypotheses and conjectures, which are then tested with further data analysis, often in collaboration with other researchers.

An important note about theory generation is that it does not require the analysis of a lot of cases since the researcher’s main goal is to develop a theory that accounts for relevant behavioral patterns. It might take one or two cases to develop conceptual categories and only another handful of cases to further examine those categories. During the research, these emergent categories pave the way for specific patterns and interrelations that guide the emerging theory.

Building grounded theory involves a progressive process of data collection, coding, analysis, and thinking about what to study next. It is not recommended for researchers to approach a grounded theory study with preconceived notions or initial conjectures of what to expect from the data. Doing so will only serve to taint the data analysis and blind the researchers of the richness of the incoming data. Glaser & Strauss (1967) contend, "In trying to reach saturation [the researcher] maximizes differences in his groups in order to maximize the varieties of data bearing on the category, and thereby develops as many diverse properties of the category as possible" (p. 62). Instead of focusing on making initial conjectures about the data, the researcher ought to focus on diversifying the character and makeup of his groups to ensure a healthy amount of variation in the data. What determines an acceptable level of saturation is up to the individual researcher to decide and depends on how broad his research goals are.

The socio-constructivist coding scheme that was used in my study developed out of an extensive review of both the classroom and non-classroom data. End-of-semester interviews were looked at prior to analysis of the videotapes of classroom sessions. This was done so that I could better trace the development of students' mathematical ideas throughout the semester. By identifying how they responded to end-of-semester interview prompts, I could go back to the classroom data and try to account for certain regularities and patterns that emerged during the interviews. I figured that it would be easier to examine the data in this way since the end-of-semester interviews had specific prompts that the students were responding to in contrast to the more free-form nature of class discussion.

Recall Figure 1 from Chapter 2 which diagrams the coding scheme used in this study. When examining the data, I tried to identify instances of horizontal and vertical maathematizing in students' reasoning as they formed chains of signification. After transcribing relevant portions of five of the end-of-semester interviews, I created a data analysis table that documented the following four issues and themes: why it made sense to set up the ratio $y/x = dy/dx$; the relationship between $y=mx$ and dy/dx; the role that proportional reasoning plays in the long-term behavior of particular solutions; and the behavior of SLSs. These four themes naturally came about through a preliminary review of the transcripts. There were several instances of horizontal mathematizing present in these transcripts which were useful for documenting advances in students' mathematical activity.

To account for the conceptual developments demonstrated by students during the interviews, the next step in the data analysis was to review transcripts of classroom data from the five days of the CTE. Specific norms of argumentation and discourse were also identified in the classroom tapes. Along with this analysis came an increase in the depth and complexity of the conceptual categories established in the interviews. The new categories were recognition and interpretation of SLSs, static and dynamic views of slope, and additive and multiplicative reasoning with the ratio y/x. Several examples of both horizontal and vertical mathematizing were extracted from the classroom data, thus establishing a link in the chain of signification.

An analysis of discussion board posts and exam data followed the classroom session research and nicely supplemented the existing data. Gradually, theories about student learning and reasoning with SLSs began to emerge from the data. These theories were modified several times through repeated runs through the video and audiotapes and discussion with fellow researchers. The result of this comprehensive analysis was a significant modification of several initial conjectures that were made about student learning and thinking following an initial run through the data. Other significant results of this grounded theory-based research were several new questions for further research and a new perspective and appreciation for conducting this type of study.

CHAPTER 4

RESULTS AND DISCUSSION

The major research questions of this study are as follows: (1) How do students construct and interpret SLSs? (2) How do students' methods of proportional reasoning contribute to their construction and use of SLSs? To address these questions, the chapter is broken down into four sections, the first of which provides a descriptive summary of the five videotaped classroom sessions. The remaining three focus on specifically addressing each of the research questions in the context of what happened during those class sessions. Specifically, the second section addresses students' recognition and interpretation of SLSs in light of the instructional activities they are presented with. The third section is centered on interpreting student conceptions of the static and dynamic qualities of the slope of SLSs as they algebraically construct SLSs. The fourth section analyzes the role proportional reasoning plays in the construction and use of SLSs with particular emphasis on the guided reinvention of SLSs using algebra. In each of these sections (except the first one) is a discussion of students' mathematization of ideas carried out across classroom activities, discussion board posts, and end-of-semester interviews.

SUMMARY OF CLASSROOM SESSIONS

Five classroom sessions are analyzed in this section. Table 1 gives a brief overview of what occurred on each of these days. Following Table 1 is a further elaboration of the events of each day.

On the first day of data collection (April 8, 2005) students were given a spring-mass problem and were asked to produce velocity vs. position graphs to describe the behavior of a mass attached to a spring with an initial condition to the right of the rest-position which represented a stretching of the spring. Students provided many interpretations ranging from graphs of sinusoidal behavior to parabolic motion on the velocity vs. position plane. Finally, everyone agreed that the graph of velocity vs. position in this situation would spiral in clockwise toward the origin.

Table 1. Brief Overview of Classroom Sessions

Day	Events	Major Themes
April 8, 2005	Introduction of spring-mass problem with corresponding set up of system of ODEs to mathematically model it Classroom exploration, involving the use of a java applet, of the effects of increasing the friction parameter on the behavior of solutions to the system of ODEs The graphical introduction of a SLS via the loss of oscillating solutions and the appearance of solutions heading toward the origin on a straight-line corresponding to a particular friction parameter value	Focus on the graphical construction of SLSs (with the aid of technology, but without mathematical proof) and physical interpretation of the behavior of points on a SLS. Prediction of position-velocity graphs.
April 11, 2005	Classroom analysis and discussion of the behavior of solutions at various initial points on the phase plane, including on a SLS Students tried to find the slope of the SLS (they did not know that there were two SLSs at this point)	Physical behavior of oscillating solutions and SLSs in terms of the original spring-mass system Discussion of how to tell analytically whether or not SLSs exist Students reasoned proportionally about the construction and meaning of slope of the SLSs (static vs. dynamic qualities of slope)
April 13, 2005	Continuation of discussion about how to find the slope of SLSs. Discussion of the number of SLSs.	More proportional reasoning about the construction and meaning of the slope of SLSs (static vs. dynamic qualities of slope)
April 15, 2005	Discussion of what the 3D graph looks like for a SLS with a given initial condition Determination of the x(t) and y(t) equations for the SLSs	Guided reinvention of SLSs using algebra

Table 1 (continued)

Day	Events	Major Themes
April 18, 2005	Description of relative behavior of two points on a SLS Analysis of exponents of SLS equations Long-term behavior of solutions not on SLSs	Focus on how students use additive or multiplicative reasoning exponents or make explicit use of a ratio in order to discuss the behavior of solutions, including those not on a straight line

Following this activity, the teacher created a second-order linear homogeneous differential equation to represent Newton's law for the sum of the forces exerted on a mass in a spring-mass situation without gravity. This second-order equation was then converted into a system of two first-order linear differential equations $\begin{array}{l} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\frac{k}{m}x - \frac{b}{m}y \end{array}$, where *k* represents the spring constant and *b* represents the friction coefficient. Next, the spring constant was fixed at 2 and the friction coefficient was allowed to vary deliberately in intervals of increase of 0.5 units from 0 to 3 using a java applet that continuously and dynamically displays the changing vector field. Many students noticed that setting the friction coefficient to 3 appears to give a graph of a solution (as viewed in the position-velocity plane) that heads directly toward the origin along a straight-line. Students were presented with a handout depicting a vector field for the $b = 3$ situation with an arrow pointing toward the origin at (-1.5, 1.5). They were then asked to figure out an algebraic way to either support or refute this conjecture, but due to time constraints they were unable to complete this task that day.

During the second day of data collection (April 11, 2005), the class continued to analyze the spring-mass problem by referring to the handout given in class on April 8, 2005. The teacher presented the class with a plot of five different points on the position-velocity plane and asked them about the behavior of solutions with initial conditions corresponding to those points. The existence of at least one SLS was presumed and the rest of class session was spent having the students try to algebraically determine what the slope of the SLSs would be and identify why the slope is an important concept. Although students were not able to correctly calculate the slope of the SLSs, there was a general agreement at the end of the class that $m = \frac{dy}{dx}$ was an important equality for finding the slope. For homework, the teacher assigned a discussion board posting about the role of proportional reasoning in identifying the slopes of SLSs to systems of ordinary differential equations.

On the third day of data collection (April 13, 2005), the class discussed the concept of proportional reasoning. The teacher asked several volunteers to explain what proportional reasoning meant to them and a debate ensued over what proportions would give the correct slope values. Agreement was reached on this issue and then the teacher asked the students whether or not they could assume that the point on the graph was even on the SLS to begin with. The goal of asking this question was to prompt students to think in a different way about how to reason with the proportions they knew. The teacher walked the students through a calculation of the slopes from here and then asked them to think about the number of SLSs. Another discussion board question was given out pertaining to how students' initial assumptions and beliefs about finding the slope of the SLS were changed by the class session or their work afterwards.

For the fourth day of data collection (April 15, 2005) the teacher began class with a recap of the previous three class sessions and explained that through algebra it was shown that there were two SLSs to the system of ODEs, one along the line y=-x and one along the line y=-2x. The students were then asked to interpret the meaning of the initial condition (-2, 4) for the same system and explain what the corresponding physical motion of the mass would be. After discussing this topic, they got into groups to analyze what the 3D graph would look like for that initial condition. Possibilities for the shape of the 3D graph were discussed as a class and then the teacher asked them to get back into their groups to determine the x(t) and y(t) equations for the solution with the initial condition (-2, 4). Through these tasks, students successfully reinvented the equations for the SLSs.

The final day of data collection (April 18, 2005) continued with further analysis of the spring-mass problem. The first question posed to students pertained to how two initial conditions on the same SLS would move relative to each other over time. Another question was asked about the importance of x(t) and y(t) having the same exponent for SLSs. Students were then asked to find the x(t) and y(t) equations for any solution with initial condition along one of the SLSs. The class session concluded with a discussion of the shape of solution graphs not on SLSs.

THE RECOGNITION AND INTERPRETATION OF SLSS

During the spring-mass investigation on April 8, 2005, several interesting comments were made pertaining to the evolution and development of SLSs. The goal of the instructional sequence that prompted students to vary the friction parameter and record their observations was that they would be able to see how solutions appeared to spiral in towards the origin more and more tightly as the friction parameter value increases. When the friction parameter nears 3, solutions appear to head toward the equilibrium solution on a straight line.

At $b = 0$, there was general agreement that solutions would oscillate back and forth indefinitely, which would be represented by closed circles on the velocity vs. position plane. While motioning to the graph of solutions on the java applet, the teacher mentioned, "See how it pushes you around in a circle and you come back." The "it" in the teacher's comment refers to the vectors that push the initial point around the plane. His use of this language of motion and movement suggests an allusion to the tip-to-tail method and the idea of vector flow. This initial discussion set the stage for the increase in the friction parameter which brought about a general agreement that solutions would spiral more tightly toward the origin.

Finally, the teacher increased the friction parameter to 3 in the java applet and placed a crosshair around the point (-2, 2). He asked students to mentally set this initial point into motion and to plot tip-to-tail graphs to see what happens to the point over time. One student, Bryce, commented that, "The friction has overcome the power of the spring, the force of the spring." It appears that Bryce is referring to the fact that solutions no longer appear to spiral – that the frictional force is now greater than the spring force and it is slowing down the solutions in a different way than was the case when the spring force was greater than the frictional force. Another student, Bobby, claimed, "We've lost our oscillating solutions." He recognized that the possible existence of a SLS is a necessary first step toward the algebraic construction of the SLS. At this point, the teacher reiterated this point and mentioned that this $b = 3$ case represented the overdamped case of spring-mass motion in physics. The teacher then mentioned, "So what…what Bryce's statement is [that] we've lost our oscillation. One way to phrase that is that our solution graph in the position-velocity plane falls along just a straight line." The goal of this statement appeared to be a pedagogical move by the teacher to facilitate further mathematization.

On the next day of class (April 11, 2005), Efrain offered his interpretation of the physical significance of the SLS by using an analogy to a traveling car. Consider the following dialogue which contains Efrain's interpretation as well as Carlos's contributions to Efrain's comments (see Appendix A – quote #1). To Efrain, the solution on the SLS with initial condition (-1.5, 1.5) behaves like a car that is slowing down due to the frictional force of the engine cutting against the forward motion of the car. He claimed that a car slamming into a wall would also exhibit the behavior of a SLS in a quicker, but equivalent way to the car slowing down to rest on its own accord. His use of the traveling car metaphor as a tool to reason about the behavior of SLSs illustrates horizontal mathematizing in the way he used a real life experience to animate his understanding of the SLS. Carlos explained Efrain's comment in a different way, stating that it was a necessary condition that forward force equaled the negative of frictional force in order for a SLS to exist. His explanation that the forces "would cause a straight line" makes it sound as though the physical conditions precede the existence of the SLS, reminiscent of what occurred during the java applet exercise.

When asked to elaborate further on the discussion of the existence of SLSs, Timothy mentioned, "So another way of saying that is the friction will slow the velocity proportionally." By "that" Timothy may mean that there exists a friction parameter that causes the mass to move toward the origin and never reach nor surpass it. Timothy's inclusion of the idea of proportionality here is of particular interest since proportional reasoning plays a significant role in determining and interpreting the slope of SLSs.

Following this discussion, the teacher had the students work in groups to try to figure out what the slope of the SLS would be. The next section provides a thorough analysis of how students conceived the static and dynamic qualities of slope and draws on data from the classroom, discussion board, final exam, and end-of-semester interviews.

STATIC AND DYNAMIC QUALITIES OF THE SLOPE OF SLSS

The major focus of this section is on how students interpret the slope of SLSs. Characteristics of a static view of slope imply that the student focuses on the slope as a numerical value, a quantity to be solved for. Under the dynamic view, students infer motion or movement via slope and offer interpretations as to how the slope of SLSs affects the behavior of solutions in the phase plane.

One point of curiosity is whether these two views, static and dynamic, are mutually exclusive or not for students. If they are then the data should show that students stick to one view or the other. If not, then perhaps there is a way that students can mathematize the meaning of slope from something static into something dynamic. If they were able to do this, then the meaning of slope to them might depend on the situation it was grounded in. In this case, they would need to have a reason for re-constructing the meaning of slope such as the demands of a new type of problem involving SLSs.

Also of particular interest in this section is how students relate different quantities, including y and x and dy/dt and dx/dt, from the system of ODEs together to reason about these slopes. These observations of proportional reasoning in action can provide useful clues to identifying the static or dynamic nature of students' understandings about slope.

For the remainder of April 11, 2005, students were asked to think about finding and interpreting the slope of the SLSs. The teacher put up a handout on the overhead projector of a point that appeared to be converging toward the origin along a straight-line. Students then got into their groups to discuss how to find the slopes of the SLSs. Li was absent from the back group on this day, so Timothy and Larry debated the issue of slope by themselves. Timothy's initial idea was to set the dx/dt and dy/dt equations equal to each other and solve for the SLS that way, as evidenced by the following two excerpts of dialogue. In the second excerpt, the teacher interacted with Timothy and Larry to listen to them justify their ideas (see Appendix A – quote #2). During the first excerpt, Timothy asserts the equality of the dx/dt and dy/dt equations to solve for the equation of the SLS. In the second excerpt, Timothy introduces the idea of a proportion between the two rates of change and says that the solution of that proportion yields a SLS. In symbolic terms, Timothy is saying that there exists a proportionality constant *k* such that $\frac{dy}{dt} = k\frac{dx}{dt}$ where $k = ½$ (it appears that to Timothy the *k* value represents the slope value); however, when he originally solved the proportion he got $k = 1$, but quickly rescinded on that value. He then concluded that the equation of the SLS was y = (½)x. When asked to interpret this SLS in light of the differential equations Larry said, "Well just think about it. Looking at this equation if holding *y* constant *x* goes up *dy* is negative so your velocity would be decreasing…same thing…well yeah." Note that Larry thought that as *x* increased, y increased as well because they made a calculational error obtaining a slope of ½ instead of the -½ they came up with later which is the correct result per their method of calculation.

After their group discussion, Timothy and Larry came up to the front of the room to discuss their ideas with the rest of the class. Timothy gave physical meaning to his idea by explaining how the change in velocity is proportional to the change in position, an idea he and Larry fleshed out in their group discussion. Timothy explained that as solutions approached the origin, both the velocity and position decreased proportionally. Consider the following dialogue in which Efrain took issue with Timothy's explanation of the mutual decrease of position and velocity over time (see Appendix A – quote #3). Efrain's confusion primarily stemmed from his confusing the dx/dt term with the dy/dt term; however, it sparked an interesting comment from Timothy who used Efrain's traveling car metaphor from earlier in class to explain the decrease in velocity over time. The use of this metaphor further suggests that Timothy, like Larry, was concerned with the effect that an action (releasing the gas petal) has on two quantities (velocity and position) in a ratio. While the velocity decreases, the position increases proportionally.

Following Efrain and Timothy's discussion about dy/dx, Carly gave her own interpretation of that quantity, saying that "dy/dx is simply the slope." For Carly, dy/dx represents the slope of the SLS – it is a fixed and static value. Earlier on in her small group discussion, Efrain asked Carly about how to find the "-2" in the "y=-2x" SLS. He found this value by looking ahead at a future activity where the slope values were known, but he was curious about how to obtain that particular value. In response to this question, Carly said, "You would basically probably have to multiply dy/dt…make it dx/dt =2*dy/dt or something like that." The way that she responds suggests that she has a calculational understanding of slope (Thompson, Phillip, Thompson, & Boyd, 1994). To her, the slope is a number to be solved for via some equation that relates the two quantities dx/dt and dy/dt.

In her first discussion board post, Carly mentioned that dy/dx can be thought of as the "change in y" over the "change in x," referring back to the proportions of basic algebra. She still did not mention anything about what the slope does or what information it provides about the behavior of other solutions and even though she refers to dy/dt as the rate of change of y and dx/dt as the rate of change of x, she does not say anything about how dy/dx itself acts as a rate of change.

During her second discussion board post, Carly reflected on the content of the April 15, 2005 lecture and made some modifications to her idea about slope (see Appendix A, quote #4).

Carly appears to have used a similar triangles argument from earlier geometry and algebra classes while reasoning about the proportionality of dy/dx to y/x. See Figure 2 for a diagram illustrating the similar triangles Carly is referring to. Note that this diagram was not drawn by Carly, but was created to assist the reader in visualizing her similar triangles explanation. It appears that this new geometrical intuition about the nature of the slope helped her to understand how to find the different slope values as well as shed some light on how her group erroneously assumed the existence of a particular proportion between dy/dt and dx/dt. What is interesting here is even though Carly had become more keenly aware of the nature of the proportional relationship between the two ratios y/x and dy/dx necessary for understanding how to find the slope, she still appeared to view slope as a static entity, evident in the way she discussed slope during her end-of-semester interview.

During the end-of-semester interview, she responded to the question about the relationship between y=mx and dy/dx, saying, “Well y=mx is the equation for a straight line and m=y/x which is the slope of the straight line.” It was clear during the interview that Carly knew what to do procedurally with the slope-first method by this point and knew that what she solved for were the slopes of the SLSs, but she still did not mention anything beyond the static nature of the slope value. Next, Anita’s conception of slope is discussed, beginning with her ideas expressed during classroom discussion and progressing to her explanations offered during the end-of-semester interviews.

Even though Anita’s calculation for slope followed essentially the same method as Carly’s group, her interpretation of the ratio dy/dx was quite different than Carly’s. When asked how she came up with her slope value, she responded with, “Well looking at it at first I found dy/dx and I thought well if it’s a line then dy/dx needs to be constant on it, right?” Notice that she reasons from the properties of the line to what the value of the slope must be in order for the idea of a SLS to make sense. In the first discussion board posting, Anita claimed, “The only proportional thing that I saw in our in-class problem is the rate of change

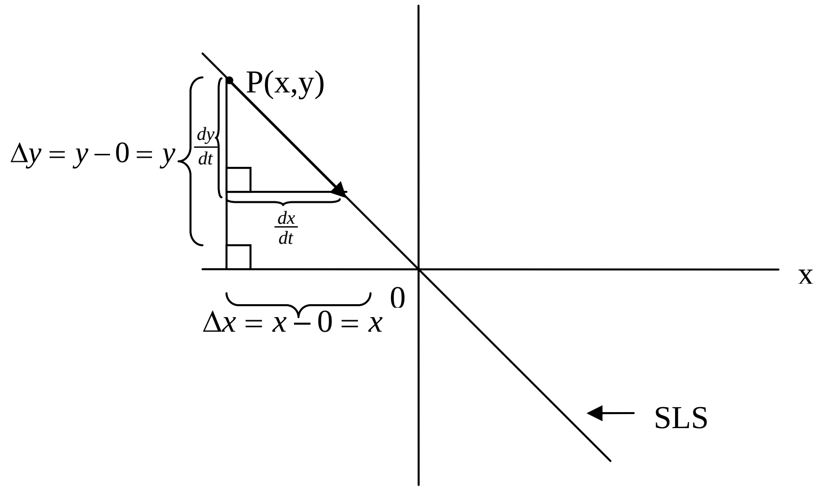

Figure 2. Diagram depicting Carly's explanation.

of y(x) which was obtained by dividing given dy/dt by dx/dt. The resulting equation dy/dx=(-2x-3y)/y gave us the symbolic representation of the slope m." Instead of using the word "slope" Anita talks about the rate of change of the SLS and explains how to find it. She also discusses the symbolic nature of the ratio of dy/dt to dx/dt.

When asked to justify the setup of her ratio m = (dy/dt)/(dx/dt) during the end-of-semester interview, Anita said, "This is a change of y on the t plane, this is a change of x on the t plane, so m is the change of y with respect to x on the y(x) plane, and it is helpful because, well, that is where we get the change of rate for y on the y, or x-y plane." In this instance, Anita connects the two individual rates of change dx/dt and dy/dt with the overall rate of change of y with respect to x. In regards to the relationship between y=mx and dy/dx, Anita commented, "y=mx and then I can cancel x, but umm the proportion of rate of change should be the same as the proportion of y/x because initially it's a slope, you get a slope." Anita equates m with dy/dx which is equated with y/x, a sign of versatile proportional reasoning among the three quantities. It appears from her reasoning that she views slope in a static manner, referring only to the fact that the slope values provide a rate of change in the phase plane. She does not mention what function or role the slope plays in terms of determining future motion or movement of solutions in the plane.

Like Anita, Timothy also expressed an understanding of the proportionality among these quantities, but the way in which he and Larry determined the slope of the SLS was peculiar. At the end of April 11, 2005, he seemed pretty confident with his work that the two ratios dx/dt and dy/dt could be set equal to each other in order to determine the value of the slope and that the value was consequently -1/2. With the guidance of the teacher and other class members, it was shown that by setting the two rates equal to each other in such a fashion, Timothy was making an implicit assumption about the ratio of dy/dt to dx/dt. While Timothy thought he was solving for the slope of the SLS by rewriting y in terms of x, what he was really doing was finding the line on which that equality occurs, if such a line were to exist, but that line is not necessarily going to be a SLS, and in this case it was not. During class on April 13, 2005, it appeared as though Timothy and Larry recognized the error in their reasoning, citing that there was no way the slope could be -1/2, but that instead it had to be equal to -1, which later turned out to be consistent with what the rest of the class obtained. In their rejection of the slope value of -1/2, it appeared as if they realized that it was incorrect to assume an initial relationship between dx/dt and dy/dt, but their reasons for choosing -1 as the slope, even though that was a correct value, were unclear and seemed to be at best an educated guess based on their observation of the vector field.

During his end-of-semester interview, Timothy made an interesting remark about the meaning of slope when asked to comment on why he set up the ratio m=-3x+4y/2x-y the way he did (see Appendix A – quote #5). Timothy interpreted m = -3x+4y/2x-y as a ratio of two rates of change (dy/dt over dx/dt). He called the slope a “changing of two axes,” likely referring to the rate of change of y over x (and implicitly relating dy/dx to y/x), an idea reminiscent of Carly’s similar triangles argument. It would appear from this dialogue that Timothy held a static view of slope, as he primarily focused on the slope as a rate of change of y over x and did not mention anything about slope as a movement or motion mechanism.

Returning to the group discussion on April 13, 2005, around the same time Timothy and Larry claimed that the slope of the SLS should be -1, Efrain came up to the front of the class to present his group’s work on this problem. He too thought that the slope would be -1, which initially elicited a general agreement from the class. Consider the following dialogue which illuminates the underlying assumptions behind Efrain’s explanation (see Appendix A – quote #6). Efrain’s reasoning seems to be grounded in his visualization about what happens to the point situated at (-1.5, 1.5). It appears as though he used the graph of the solution at this initial point to conclude that in order for the point to travel toward the origin it would need to have a decrease in velocity with a proportional increase in position such that the proportionality constant between y and x was exactly -1. Even though Efrain correctly produced one of the slope values, his path of reasoning to get there was circular. Kathryn pointed this out in the way that he already assumed a certain relationship existed between dx/dt and dy/dt and then used that to prove that the slope was -1.

The following response to Kathryn's criticism from Efrain makes it clear why he reasoned the way he did about the slope (see Appendix A – quote #7). It seems as though Efrain's reasoning partially lies in the design of the worksheet, which was designed with a java applet with crosshairs centered near the point (-1.5, 1.5). Even though the coordinates of the point represented by the crosshairs were not exact, Efrain and several other students thought that they could just assume that this point was on the SLS. With that assumption, it would be easy to find the slope of the SLS using the method of finding the slope of a line from elementary algebra since they would have the two points necessary for determining the slope of a line, namely (-1.5, 1.5) and (0,0). According to another student, Laura, "A lot of us are caught up on the assumption that this specific point, point #6, is on a SLS, but we don't know if it is." In retrospect, it would have probably been more effective from an instructional standpoint not to explicitly label the point in question nor set up the problem to have a SLS with a slope of -1.

In his second discussion board post, Efrain mentioned, "We cannot make assumptions because we want to prove the slope for any given condition, which we cannot do if we assume that the slope is one already." It is clear from this post that Efrain was aware that he used circular reasoning to conclude that the slope was -1. Despite the point labeling problem on the worksheet which may have confused Efrain to begin with, it seems as if he made an important realization about the nature of his reasoning. His circular reasoning was grounded in the way he reasoned about the ratio of dy/dt and dx/dt as separate quantities from m. It was not until after the group discussion that Efrain recognized the way that making an initial assumption about the quantity of the ratio of dy/dt to dx/dt was exactly the same idea as assuming the value of m.

Although Mario did not actively participate in class or group discussion on April 11, 2005 or April 13, 2005 (nor did he complete the first or second discussion board questions), his comments about slope and proportionality during the end-of-semester interviews were insightful. When asked about why the proportion y/x = 3x+4y/2x-y makes sense to him, Mario had the following to say (see Appendix A – quote #8). Mario explains that the idea of proportionality "makes sense" since the SLSs have to be linearly independent in order for them to serve as a basis for other solutions in the phase plane. He refers to the slopes as "bases," but it seems likely that Mario is using "slope" as a metonymy for the straight-line. In this way, he views the SLSs as building blocks for all other solutions in the phase plane. This idea suggests that Mario has a dynamic, global view of the SLS. More evidence for this claim about Mario's view of slope can be found in the way he answers the question about the relationship between y=mx and dy/dx. Consider the following dialogue between Mario and the interviewer (see Appendix A – quote #9). Mario first explains that the ratio of dy/dt to dx/dt has to equal that of y to x in order for a constant slope to exist. He then makes an interesting comment about the rate of change being "the future of y/x," which appears to indicate how Mario horizontally mathematizes the slope of individual vectors into the slope of the SLS. This idea of the slope of individual vectors predicting the slope of the entire SLS moves well beyond the static conception of slope and presents the rate of change as a dynamic tool that can be used to make inferences about the slope of the SLS via the unitizing of the slope of individual vectors.

Similar to Mario, Li was absent from class on April 11, 2005 and did not actively participate in class discussion on April 13, 2005 (nor did she complete the first or second discussion board questions); however, she did provide useful data during the end-of-semester interviews. In response to questions about the slope-first method, Li first explained that she set up the equation dy/dx=-3x+4y/2x-y=m like that because "if we're looking at shapes in the plane then it only makes sense to look how y and x change with each other." She claimed that the only way to make sense of the slope of the curves in the plane is to look at the ratio of dy/dt to dx/dt since it is the rise over run.

To her, the "-3x" term reminds her of the rate of change present in this ratio and it is "built into these equations." She commented that y and x have some kind of dependency on each other, but she does not further elaborate on that dependency except to say that she could have used the eigenvalue-first method and set up a matrix of coefficients and figured out what the dependencies were. Concerning the dy/dx term, Li explained, “It’s just a change showing how y changes over how x changes or conversely, how y changes as x changes.” Li’s interpretation of the dy/dx term appears to be static since she does not mention anything beyond slope as a rate of change.

Further evidence of Li’s static interpretation of slope is present in the following dialogue, a response to the question about the relationship between y=mx and dy/dx (see Appendix A – quote #10). When asked about the relationship between y=mx and dy/dx, Li made an interesting comment that “for me it’s almost like two halves of a coin.” She claims that finding out what the slope is determines the dependencies between y and x. There is a lot of proportional reasoning going on between m, y/x, and dy/dx, evident by her responses to the interviewer’s questions, but her interpretation of slope appears to be static in nature in the way she refers to it as a constant that determines the dependencies between two quantities and not as a vehicle for motion or movement in the plane.

Of the twenty-five students surveyed in the first discussion board posting, twenty three felt that proportional reasoning was helpful in identifying the slopes of SLSs. Another student felt it was not helpful and yet another student did not discuss it at all, making it unclear whether or not he felt it was useful. The student, Mia, who did not feel it was helpful had the following to say regarding proportional reasoning (see Appendix A – quote #11). She referred back to an earlier task done in class where individual slope values were calculated by plugging particular values into the dx/dt and dy/dt equations and then their ratio was taken using the fact that m = dy/dx. However, she realized that using this technique will not work when a nontrivial point that passes through the SLS is unknown, as is usually the case. To find the slope, substitution of y=mx is used into the ratio dy/dx. Even though this process definitely involves proportional reasoning, it seems as though Mia regards it as using “past techniques.” She does not mention where these techniques came from, although she may be referring to the earlier method of plugging in points into the bees and flowers example.
The students who found proportional reasoning to be helpful in identifying the slope of SLSs had one major point of commonality throughout their posts, namely that the use of proportions enabled them to make connections between the different ratios algebraically and graphically. Consider the following post from Kathryn, a student who described this

connection in more detail (see Appendix A – quote #12). Kathryn begins by defining the slope of a line as a rate of change of rise over run. Then she uses an example of a line where the slope between any two points remains constant. She concludes by mentioning that being able to figure out the proportion of the rate of change of dx/dt and dy/dt algebraically enables one to determine the slope of SLSs. Many other students made this connection between their prior knowledge of rise over run from high school algebra and how to work with the ratio of dy/dt to dx/dt. Bryce, another student with a similar idea about slope, phrases his explanation in the following way (see Appendix A – quote #13). Bryce's idea of the slope as a proportionality constant helps him to understand the quantities that are being compared, namely y and x and dy/dt and dx/dt. What is implicit in Bryce's post is that since the slope is constant on the line, this proportionality constant can be set equal to y/x and due to the similar triangles proportion, m can be set equal to dy/dt over dx/dt. Again, reference Figure 1 for an illustration of this comparison.

The next discussion board question was posed following class on April 13, 2005 and was focused on whether or not students' understandings of how to find the slope of SLSs changed following that class session. Of the twenty-six students who responded to the prompt, twenty-four reported a significant change in their initial understanding while two did not report any significant change. Those who did not report any change did so because they had trouble following what was going on during class on April 13, 2005, attributing their confusion to issues surrounding the exact location of the point on the java applet handout. By far, the most significant realization among the students seemed to be their awareness of the dangers of falling into circular reasoning about the slope of SLSs by assuming something at the outset that is later tried to be proven. These remarks indicated that the students not only learned something about finding the slope of SLSs, but about the nature of proof as well. Consider the following comment by Hopi, who demonstrated her awareness of the dangers of circular reasoning (see Appendix A – quote #14). Before Wednesday's class Hopi, like many others, thought that y=-x was the equation of one of the SLSs, but was not completely sure why because of the lack of algebraic proof. After April 13, 2005, however, she gained a clearer understanding of the relevant proportional relationships and felt satisfied that there were two SLSs, confirming her initial suspicions that there would be at least one SLS with a negative slope. One visual clue for her was that she knew the point labeled on the SLS would have to get "pulled into the center." This comment probably has to do with the fact that the point in question was in the upper-left portion of the Cartesian plane and because the slope values appeared to be negative there, but she had no way to prove the existence of the SLS y=-x at this point.

Hopi also commented at the initial surprise at finding that there were two SLSs as opposed to a single SLS. For her, the algebraic proof of two SLSs was satisfying because it finally allowed her to definitively determine the slope of SLSs. Like her, many of her classmates also found the fact that there were two SLSs to be surprising at first, but they were able to rationalize its existence with the methods of elementary algebra. According to another student, Samantha, "Because we use algebra to find the slope we found that there are more than one SLS, which tells us that our assumption are not always correct and also the graph might not show all solutions because on the graph we didn't now see two straight line solution." Samantha notes that by just looking at the graph of the vector field it can be very difficult to locate exactly where SLSs exist and it can be even more difficult to try to discern their slopes from the graph. With the algebra, however, there is definitive, convincing proof that these SLSs exist and in some cases, as with this particular problem, the results can be quite surprising.

In the course of conducting this study, several initial hypotheses were altered significantly. Initially, it was thought that a static approach to slope was inferior in its complexity to a dynamic approach and steps ought to be taken in instructional design to encourage the development of a more dynamic, evolving view of slope. The reason for this line of thinking was that students holding the dynamic view seemed to use the slope of the SLS as a tool for reasoning more broadly and generally about the behavior of other solutions in the phase plane. The slope under this view was seen as a vehicle for motion and movement and not merely a quantity to be solved for. Surely there could be no more sophisticated argument than the one given by Mario in which he described the slope as the "future of y/x."

After reviewing the data several times, it became clear that this bi-polar view of slope was flawed. Instead, it appeared that students developed sophisticated models for the meaning of slope even under the static viewpoint. For instance, consider Li's description of y=mx and dy/dx as "two sides of a coin" or Carly's sophisticated similar triangles argument.

These ideas were just as well developed as Mario's and offered just as much insight into the nature and meaning of the slope of SLSs. Of the five major cases studied, four students seemed to view slope as a static entity and only one seemed to view it as a dynamic entity. Although the sample size is relatively small, it is clear from the data that there are many sophisticated ways to think about slope without thinking about its motion or movement.

It also appears from the data that there is little movement between the two views of slope and that there is no evidence to support a progressive mathematization from one view to the other. This does not mean that no such links exist, but as of yet, there is no data that supports such a link from static to dynamic or vice-versa. One possible explanation for this finding is something akin to the Heisenberg Uncertainty principle from physics, which states that one cannot know the exact position or momentum simultaneously of a particle moving in space. Perhaps it is this way with slope too. If one, such as Mario, focuses on the way in which the slope of individual vectors predicts the slope of the SLS, then it might be difficult for him to comprehend slope as a static rate-of-change entity without abandoning his notion of motion and movement.

Another possibility is that the terms "static" and "dynamic" are not yet well-defined and further revisions need to be made to the definitions. The reason for this thought has to do with the wide variations in the type of "static" views of slope. It was initially thought that for a student to possess a static view of slope, they would merely identify the value of the slope and recognize it as a rate of change. Looking at the data, this is clearly not the case. Another alternative, if one accepts the distinction in terms, would be to revise the static view entirely.

One suggestion would be to break the "static" term up into two new terms. The first term could be dubbed the "calculational" view of slope and could be used to describe instances in which the slope is merely viewed as a quantity to be solved for or a simple rate of change. The second term could be called the "proportional" view of slope and would describe views of slope that take into account proportional relationships such as Carly's similar triangles argument or any other description involving a comparison of ratios. With these two new terms, one can see the potential emergence of a horizontal continuum among the calculational, proportional, and dynamic views of slope.

Ultimately, further research in this area is necessary to further clarify these distinctions and to either support or refute the present conjecture that these two views are mutually exclusive.

RE-INVENTION AND USE OF SLS EQUATIONS

The purpose of this section is to illustrate students' guided reinvention of SLSs, focusing on the development of the x(t) and y(t) equations for the SLSs. Subsequently, students used these equations to make inferences about the long-term behavior of any solution, including those not on a SLS, in the phase plane.

One point of curiosity is how students initially arrived at the x(t) and y(t) equations for the SLSs. Also of interest is the extent to which students used proportional reasoning with these equations to make sense out of the long-term behavior of solutions. When proportional reasoning about a ratio of solutions was not explicitly used, students used additive and multiplicative reasoning to determine the behavior of solutions not on a straight line. These latter methods of reasoning involved an analysis of the exponents of the SLS equations. It would be interesting to examine how often each of these different methods of reasoning about the behavior of particular solutions was used and if there was any advantage or disadvantage to one over the other.

At the beginning of class on April 15, 2005, students were asked to determine what the three-dimensional graphs of the y=-x and y=-2x SLSs that were solved for on the previous day looked like. One pedagogical goal of this activity was to transition students into constructing the x(t) and y(t) equations for the SLSs so that they could find the general solution to the system of ODEs. To make this transition more meaningful, the teacher mentioned that just as it was just as mathematically and physically significant to determine the x(t) and y(t) SLS equations as it was to find the slopes of the SLSs.

Three major conjectures emerged in the course of class discussion as to what the shape of the SLS in three-dimensions would look like. One student, Danny, described the behavior of the SLS as being "linear towards the origin." In this view, he envisioned a line that came out toward the t-axis. When the teacher called for a show of hands of students that were in agreement, the majority of the class raised their hands. Danny's reasoning about the shape of the SLS in this way made sense given its appearance on the phase plane.

Immediately following his initial idea about the behavior of the SLS in three dimensions, Danny offered up another conjecture. Alternatively, he claimed that it was possible that the SLS behaved as a “decreasing exponential [that] crosses the origin.” When the teacher called for a show of hands this time, few students raised their hands in agreement. Immediately following the show of hands, another student, Maya, claimed that a decreasing exponential could never cross the origin. This comment brought about the third conjecture about the three-dimensional behavior of the SLS – it acted as a decreasing exponential which approached the origin asymptotically.

Earlier in group discussion, Li made a similar series of conjectures to Danny, at first believing that the graph would behave as a line heading toward the origin. Consider the following dialogue (see Appendix A – quote #15). After listening to other people talk about the SLS behaving like a decreasing exponential, Li changed her mind. Her explanation had to do with the way x(t) and y(t) would have to change in time, possibly referring back to earlier work with the solutions of one-dimensional ODEs.

Li then offered up a metaphor to further explain the decreasing exponential behavior (see Appendix A – quote #16). Her closing door metaphor offers an excellent way to think about the behavior of the SLS in three dimensions. Just as a door operating on a hydraulic lever, points on a SLS slow down very quickly at first, but then with increased resistance, they begin to approach rest more slowly. Since this is the way that the mass-spring system operates under the force of friction, it would make sense that the SLS would decrease rapidly at first and then slow down in time due to the resistive force of friction.

Back to class discussion, the teacher appeared to be satisfied with the development of these three conjectures, evident by the following comments (see Appendix A, quote #17). At no point in his commentary did the teacher offer his opinion as to which view is the mathematically correct one. This decision most likely served as a pedagogical tool for advancing the classroom mathematical activity toward finding the x(t) and y(t) equations so they could reconcile which of the three interpretations for the three-dimensional behavior for the SLS was correct. With this move, students had natural motivation to try to find the x(t) and y(t) SLS equations since doing so would provide them with the answer to the previous question about the shape of the solutions.

After several minutes of group discussion and debate, the front group came up with a way to find the x(t) and y(t) equations (see Appendix A – quote #18). The initial thinking from Efrain's group was to substitute y=-2x into the dx/dt and dy/dt equations to get each strictly in terms of x and y, respectively, and then to solve for x(t) and y(t) using separation of variables. This method is outlined as follows: substituting y=-2x into $\frac{dx}{dt} = y$ yields $\frac{dx}{dt} = -2x$. Using separation of variables yields $x(t) = k_1 e^{-2t}$. Similarly, the same method can be used to solve $\frac{dy}{dt} = -2x - 3y$ for y(t). In summary, these students were able to apply their knowledge of how to solve first-order ODEs along with a clever substitution to solve for the x(t) and y(t) equations to this particular system. Notice that this method only relies on these students' basic algebra knowledge and previous experience with the separation of variables technique. They successfully algorithmatized their experiences into a general method that could solve for the x(t) and y(t) equations for any linear system. Li's group also came up with the same substitution method for finding the SLS equations. The fact that two groups independently arrived at these equations strongly suggests a strong intuitive base to these methods.

During the class discussion that followed, there was some confusion among classmates as to how to find the x(t) and y(t) equations. Timothy, who was working independently of Li and Larry who successfully derived the equations, experienced difficulty with trying to solve for these equations. He just tried to integrate the dx/dt and dy/dt equations immediately without changing variables and got confused. Consider the following dialogue with the camera-man, in which Timothy explains his reasoning about this problem (see Appendix A, quote #19). Timothy was not the only student in the class who thought he could integrate the dx/dt equation with respect to both y and x. Students who think that they can perform a single integral with respect to two variables may do so either because of their unfamiliarity with multivariable calculus or because they just do not see the need to substitute in the SLS equation like Li did.

The teacher addressed these types of difficulties by saying, "I think on the second day of class I said part of the difficulty in differential equations is that you have to think about x as both a variable in differential equations a variable but it's also an unknown function." This comment reflects the dynamic nature of x. In the differential equations themselves, x acts as a variable, but in solving for the equations for the SLSs, it acts as a function of time. Students must understand both aspects of x and y in order to be able to solve for the x(t) and y(t) equations.

Even after this explanation of x as both variable and function, there was still some confusion among the students about how they could just plug in the y=-2x SLS equation into the dx/dt and dy/dt equations in order to solve for the x(t) and y(t) equations. Having already figured it out in his small group, Mario offered up the following explanation, which the teacher then summarized (see Appendix A – quote #20). Mario's comment here is reminiscent of his unitizing of the slope of the vectors that make up the SLS as "the future of y/x." The teacher capitalizes on this comment, mentioning that even though there are many reasons why one can substitute in y=-2x to dx/dt, one of the most convincing reasons is that the SLS y=-2x is made up of an infinite collection of vectors with the same slope, so that no matter what initial condition is used, so long as it lies on one of those vectors, then that substitution can be used.

Near the end of class, the teacher brought back the question to the class about what the three-dimensional graph of the solutions would look like. The following dialogue illustrates this discussion (see Appendix A – quote #21). Notice how the teacher immediately segued from discussion about the x(t) and y(t) equations into talk about the shape of the solutions in three-dimensions. When he asked the class if they still thought that the equations would cross the origin, there was a resounding "no" from the class with Maya being one of the most vocal respondents. Using the exponential nature of the x(t) and y(t) equations, the teacher agreed, explaining that it would be impossible for an exponential solution to ever cross the origin. Therefore possibility #3, initially posed by Maya, that the SLS would exponentially decrease toward the origin turned out to be the correct conjecture.

In her group, Carly also recalled that when she and Efrain experimented with the java applet at home earlier in the week, they found that the SLSs appeared to behave exponentially, but they did not know why this was the case. With the algebraic evidence in hand from class discussion, Carly expressed confidence about her findings on the java applet. Her use of technology allowed her to anticipate what the 3-D graphs of solutions would look like which might partially account for how her group initially thought about how to find the x(t) and y(t) equations.

In summary, it appears that the intent of the activity to have students figure out what the x(t) and y(t) equations were for the SLSs and be able to subsequently visualize the behavior of these solutions in three-dimensions was successful. By having the students first hypothesize what the three-dimensional graphs looked like, they were motivated to find the equations to test their initial hypotheses. At the end of class students figured out the answers to both of these questions and best of all, expressed a genuine sense of ownership over their ideas.

At the beginning of class on April 18, 2005, the teacher posed a question about what the relative behavior of two paths traced out from two different initial conditions along the same SLS would look like. He placed a transparency on the overhead projector depicting these two different initial conditions, one in blue and another in green. The blue one was furthest from the origin and the green one was closest to the origin. Larry came up to the front of the room to trace out the relative behavior of the two solutions in time. His initial idea was that they would maintain the same distance all the way down the line until the green one touched the origin, at which point the blue one would eventually catch up to it.

Roberto disagreed with Larry's idea and said that they would not maintain the same distance, but instead the blue one would gradually start to catch up with the green one over time. Efrain then asserted that the idea of one catching up with the other did not make sense at all because the slope of the SLS was constant. At this point, several students raised their hands and debated over what would happen (see Appendix A – quote #22). To answer Efrain's question about how two points traveling along the same line with the same slope could possibly overtake one-another, Timothy explains the exponential decaying nature of the x(t) and y(t) solutions. The further out the point is to begin with, the faster it will initially travel before it begins to slow down near the origin. His idea is very similar to Li's closing door metaphor presented on April 15, 2005 in which she explained that exponential decay functions start out with very rapid decay but then gradually slow down as they approach rest. From Timothy and Mario's viewpoint, the same principle appears to hold for solutions on the SLS. Following this discussion, the teacher asserted, "These are the same graphs – one is just shifted to the other in 3-space you get that multiple shifting just like our case for autonomous differential equations for a single d.e. In 3-D we have shifts of each other along the t-axis." What he was saying is that so long as the point chosen lies on the same SLS as other points, then they just represent a shift along the t-axis relative to one-another. This explanation is consistent with Timothy and Mario's claims that both solutions would asymptotically approach zero at the same rate, but the distance between the two points would gradually decrease over time, never reaching zero, because of the relative speeds of the two vectors. Furthermore, the teacher was able to relate what they were doing to an activity done earlier with one-dimensional differential equations.

Following a few minutes of group discussion, the teacher returned with a useful metaphor for thinking about the exponential term in the x(t) and y(t) equations (see Appendix A – quote #23). Thinking about the exponential term as a speed controller makes sense out of the slowing down of points along the SLS over time. This metaphor reflects vertical mathematizing with the x(t) and y(t) equations in that they are now being used to form conclusions about the relative velocity of two points traveling on the same SLS. The creation of this metaphor ties in quite nicely with the next question posed by the teacher which deals with the homogeneity of the exponents of the SLSs. He wanted to know why it was essential for both the x(t) and y(t) equations to have the same exponent in order for the SLS to exist.

To answer this question in his small group, Mario referred back to his proportional reasoning from April 13, 2005. According to him, "It would make sense that there would have to be a similar exponent. That way you know you could actually do a proportion between x and y." Mario is able to reflect on his previous experience working with the ratio of y to x in order to reason about why the ratio of the y(t) to x(t) equations would have to be constant. Following group discussion, the teacher explained that, "What we can see over here as well is that this is just a fraction of the ratio 4 to -2. A fraction of the ratio 4 to -2. So having this same exponent is critical and essential to being able to maintain the same ratio." Notice how the teacher calls attention to the same ratio holding on the SLS no matter where the points are located. He does this in a clever way, citing the phenomenon as being a fraction of the original, but equivalent, ratio. So no matter what two points one selects as initial conditions on the SLS, each of them must always maintain the same ratio of y to x as they move along the straight-line. This elaboration further clarifies the necessity of being able to view x and y both as a function and variable.

By the end of the semester, students developed a sophisticated understanding of the x(t) and y(t) equations to the point where they could use them as tools to reason about other situations related to systems of ODEs. One such example of this application of these equations as tools for further reasoning occurred during the end-of-semester interviews where students were asked to make inferences about the long-term behavior of solutions not on SLSs. There were two main types of reasoning employed to answer this question: additive and multiplicative reasoning. Students reasoning additively employed a “force” or “pull” argument to account for the movement of solutions toward one SLS or the other. Multiplicative reasoning, on the other hand, could be classified by students making explicit use of the ratio of y/x in their argument.

Timothy makes implicit use of ratio in his proportional reasoning about the long-term behavior of solutions when he claims that SLSs with bigger exponents will have a greater pull on particular solutions than SLSs with smaller exponents. Consider the following dialogue where Timothy explains the pulling effect of the SLSs on solutions that do not lie on a straight line (see Appendix A – quote #24). Technically speaking, the eigenvector is any vector that lies along the SLS and the eigenvalue is the exponent of the x(t) and y(t) equations for that particular SLS. By this time in the interview, Timothy had already formed the general solution to the system of ODEs. Before sketching the phase portrait, he decided to check his previous work with the eigenvalue-first method. He sketched the phase portrait using the relative pulling power of the eigenvalues. When the interviewer probed deeper into how he came up with the shape of particular solutions, Timothy mentioned something about limits, saying that “this term will uh not be able to compete with this term…because exponent e^5t is a multiple of 5…is significantly bigger, so this term will be dominating.” His language about the exponent dominating the behavior of solutions going forward in time is indicative of additive reasoning with the exponents and simultaneously, vertical mathematizing with the x(t) and y(t) equations. As time goes on, the particular solution given by an initial condition approaches the SLS with greater eigenvalue because of the cumulative pulling power of that SLS.

Another student who reasoned additively about the behavior of initial conditions not on the SLS was Li. The following dialogue illuminates her reasoning (see Appendix A – quote #25). Li uses a force argument to describe the behavior of particular solutions backward in time in the phase plane relative to the “pulling” power of the two distinct SLSs. She initially uses $e^{-5\infty}$ to describe the greater pulling power of the y=-5x SLS in opposition to the weaker y=x SLS. Later on she mentions that "And then as they go to positive infinity, they're gonna have slopes closer to y equals minus 3x." As with Timothy, Li expressed a certain accumulating nature to the slope that particular solutions take on over time.

Unlike Li and Timothy, Mario made use of multiplicative reasoning in describing the behavior of solutions not on a straight line. Consider the following excerpt from his end-of-semester interview in which he explains his reasoning (see Appendix A – quote #26). Notice how Mario makes explicit use of the ratio y(t)/x(t) in his analysis. By taking the limit of this ratio, he is able to algebraically determine which SLS a particular solution in the phase plane approaches as time increased in the positive or negative direction. This use of ratio illustrates an example of vertically mathematizing with the x(t) and y(t) equations to make a conclusion about the behavior of particular solutions in the phase plane and subsequently allows students to easily sketch the phase portrait for the system of ODEs.

Anita also made explicit use of ratio in her analysis of the long-term behavior of particular solutions in the phase plane, evident by the following quote (see Appendix A – quote #27). Just like Mario, Anita took the limit of the ratio of y/x to observe what would happen to initial conditions not on a straight-line. Her statement about the graphs "approaching both time periods" shows how she views the solutions evolving in forward and backwards time with these limits.

Like Anita, Carly also took the limit of the ratio of y/x. She also described the nature and character of the slope as it applies to particular solutions as follows (see Appendix A – quote #28). Notice how Carly talks about the limits telling her what the slope is going to look like for the non SLSs. Later on in the interview she develops the language of limiting slope to describe this long-term behavior (see Appendix A – quote #29). Her idea of a limiting slope for solutions not on a straight line reflects the notion that any particular solution in the phase plane will approach the slope value of one SLS or the other without ever exceeding it. In creating this term, it appears that Carly may have algorithmatized earlier activities involving the slopes of SLSs and her experiences with limits in calculus. Since this term evolved naturally through Carly's mathematical activity, it would be reasonable to expect that other students, who may have reasoned additively about the long-term behavior of solutions, would be able to understand the idea of limiting slope and apply it to other problems.

There were many instances throughout the semester of students re-inventing solution methods through mathematical activity and then sharing their methods with other classmates. One prominent example occurred during the beginning of the semester when the teacher introduced the students to vector flows in the plane. The students were asked to make predictions about the future behavior of populations modeled by a first-order ODE without having access to any explicit solution methods. During the course of this activity, several students actually re-invented their own rendition of Euler's "tip-to-tail" formula without realizing it until the teacher made them aware of it. These students developed a sense of ownership over their ideas and fellow classmates had little difficulty understanding them since the formula was in a language familiar to the students. Similarly, students such as Carly were able to develop formal mathematics through progressive activities involving the expected development of certain concepts. In many cases, the ideas developed such as Carly's idea of limiting slope paved the way for better conceptualization of important mathematical ideas and greater computational and analytical fluidity with problem solving.

There was a question on the final exam concerning students' abilities to apply the ideas of additive and multiplicative reasoning toward sketching graphs of solutions to a system of two ODEs in the phase plane. The students were given the general form of the system as well as the general solution. The equilibrium solution (0,0) turned out to be a sink, meaning that all solutions approached the origin exponentially in time. Following a sketch of these solutions, they were asked to justify why the solutions that do not lie along straight lines in the phase plane look the way they do as t goes to positive infinity and negative infinity, respectively. Of the thirty-seven students who took the final, twenty-three used additive reasoning, six used multiplicative reasoning, one used both additive and multiplicative reasoning, and seven used a form of reasoning that appeared to be neither additive nor multiplicative in nature. Of the twenty-three students who used additive reasoning, nine students (39%) received full credit and fourteen (61%) did not. Of the six students who used multiplicative reasoning, two (33%) received full credit and four (67%) did not. The student who employed both methods received full credit and of the seven students who used a non-additive and non-multiplicative form of reasoning, zero (0%) received full credit and seven (100%) did not.

These results were surprising, as it was initially conjectured that students who used multiplicative reasoning would perform significantly better than those who used additive reasoning. One reason for this type of thinking was that the use of an explicit ratio, such as Carly's limiting slope, seemed to be a direct and straightforward calculation whereas additive reasoning involved a less rigorous examination of the eigenvalues which appeared to be easily subject to error, especially due to the wide variety of possible equilibrium solutions. It turned out that students who used additive reasoning to graph particular solutions performed slightly better than those who used multiplicative reasoning. As expected, students who used a non-additive or non-multiplicative line of reasoning performed the worst of all.

One possible reason to account for these results is that the limit calculations involved in computing the slope of the SLS that solutions approach might actually be more complicated than initially thought. In the course of taking the limit of the ratio of y/x, students must carefully manipulate exponential expressions so they do not end up with an indeterminate form. Also, this process can be quite time-consuming since two limits must be computed, especially if the student does not initially recognize that it is only really necessary to compute one limit.

Students who did not use either form of reasoning, multiplicative or additive, appeared to lack a basic understanding of the concepts involved in identifying the shape of the graphs of particular solutions. A few of them appealed to methods of reasoning reminiscent of a "force" or "pulling" argument, but it was so incomplete and wrought with errors that it was difficult to tell if that was what they were really doing or if they were just blindly guessing.

It does not appear that there is an advantage to using one method of reasoning over another. Although there was slightly better performance with students who used additive reasoning, the percentages were not different enough to warrant the exclusion of multiplicative reasoning or strict inclusion of additive reasoning techniques in a differential equations curriculum. As shown with the one student who used both methods and received full credit, perhaps it is best if students have an awareness of both multiplicative and additive forms of reasoning so that they can decide which might be more efficient to use with different problems.

Another initial conjecture that was changed was that these two methods would function independently of each other, much like the static and dynamic views appeared to act. It turned out that there were a few students who used additive reasoning in the end-of-semester interviews who used multiplicative reasoning during the final.

It is truly amazing how these students were able to progress from the re-invention of the SLS equations toward generalizing about the shape and long-term behavior of any solution in the phase plane. What began as an activity involving the three-dimensional visualization of the SLSs created during the previous class sessions turned into a powerful display of vertical mathematizing with the x(t) and y(t) equations. Not only did the students demonstrate remarkable enthusiasm during these classroom activities, but they also successfully achieved conceptual development above and beyond that originally anticipated by the teacher. As a result of this sophisticated classroom interaction, new ideas and methods have been developed that can be used to guide future teaching and learning in differential equations.

CHAPTER 5

SUMMARY, CONCLUSIONS, AND RECOMMENDATIONS

This chapter begins with a brief summary of the entire research effort, including an overview of the conceptual framework, design of the investigation, methodology, and the results of the study. This section includes a discussion on the significance of the study and its conclusions. The second section is focused on the limitations and weaknesses of the research and the last section provides recommendations for teaching and further research.

Brief Summary of Study

The main purpose of this study is to document and analyze how students construct and conceptualize SLSs to systems of linear ODEs. From a dynamical systems perspective, systems are an important way to describe and model mathematical situations involving rates of change. SLSs are mathematically significant ideas because they form the basis for all other solutions in the phase plane. A second objective of this study was to examine the role proportional reasoning plays in the construction and interpretation of SLSs. Lastly, this study examines implications for further teaching and research.

The conceptual basis for this study is rooted in RME-based instructional design theory. One of the key features of RME is the belief that students can re-invent mathematical knowledge for themselves through realistic mathematics activities. The goal is for students to be able to develop models for mathematical reasoning through experientially real problem-solving situations. Rasmussen et al.'s (2005) theory of advancing mathematical activity describes the learning process as being defined by ongoing participation and practice in a community of learning. Instead of students acting as blank slates and collecting finite units of knowledge, they engage in the dynamic, evolving practice of knowing. Through this practice, students are able to horizontally and vertically mathematize their experiences with the creation of meaningful artifacts, symbols, and algorithms that build toward the development of progressively more formal mathematics.

The theoretical orientation for this study is derived from a socio-constructivist perspective on learning whereby the meanings and practices of a community of learners evolve from the activities of individuals participating in this community (Cobb & Yackel, 1996; Sfard, 1998). The instructor works with the students to negotiate certain social and sociomathematical norms that will be beneficial to both the goals of the researcher and students. These norms develop through the daily interactions in the classroom between the teacher, students, and mathematical activities, inclusively. The interaction of individual activity and conceptions against the background of these norms gives rise to the creation of mathematical models for thinking and reasoning. These models can increase in sophistication and complexity through chains of signification, which act as ongoing interactions between the acquisition metaphor and participation metaphor. These chains are the primary vehicles for the mathematization of classroom experiences into formal mathematical knowledge.

As the appropriate social and sociomathematical norms come into place, the classroom teaching experiment (CTE) can be successfully conducted. The social norms that developed in this study included ones that enabled open discussion among students and teachers concerning their mathematical ideas and justifications. For instance, the teacher was not seen as an authority figure, but instead was viewed as a negotiator and moderator of mathematical meaning. The nature of mathematical activity and discourse that developed encouraged students to take ownership over their ideas and to feel comfortable in sharing them with the rest of the class. There were several documented instances of classroom discussion where students referred to each others' existing ideas to explain new ones.

Interaction analysis and grounded theory provide the theoretical background for the methods of data collection and data analysis used in this study, respectively. The use of videotapes, audiotapes, and the ongoing collection of written work allowed for a fluid body of data that could be repeatedly examined to identify certain patterns and regularities in students' mathematical claims and justifications. Routine de-briefing sessions allowed for the exchange of ideas among researchers and the ability to modify lesson plans in a way that would move their mathematical agenda forward. Homework assignments, discussion board postings, and exams were also collected and examined to record how students conceptualized and discussed problems involving the slope and proportional reasoning with SLSs. End-of-semester interviews were used to observe how these ideas with SLSs evolved in sophistication and complexity over the course of the semester.

The CTE for this particular study spanned five consecutive instructional days, two of which were taught by me. The overarching pedagogical goal during these sessions was to guide students toward re-inventing, interpreting, and using the equations for the SLSs to the system of ODEs that mathematically model the spring-mass problem. During the first day of instruction, students were introduced to the spring-mass problem and used an interactive java applet to vary the friction parameter of the system. This activity led to their recognition of the SLS as they observed spiraling solutions lose their oscillations. The next two days were devoted to discussing the slope of SLSs and the role proportional reasoning plays in determining the slope. They also reasoned about the corresponding physical behavior of solutions given any initial condition in the plane. On the fourth day, students were asked to determine what the three-dimensional graphs of SLSs looked like in space. To answer this question, they found the x(t) and y(t) equations for the SLSs. On the final day of the CTE, they used these x(t) and y(t) equations to reason about the long-term behavior of any solution in the phase plane, including those not on a straight line.

During the end-of-semester interviews, students were presented with two tasks. The first task presented a system of equations with a saddle node equilibrium point. Students were asked to solve the system using both the slope-first method and the eigenvalue-first method and then to comment on the effectiveness and computational efficiency of each method. The second task, although not used in my CTE, focused on how students approached novel problems involving the slope of SLSs.

One initial finding of this study was that there were two primary ways students viewed the slope of SLSs: static and dynamic. Initially, it was thought that the static view would be an impoverished view with students only focusing on the slope as a value to be solved for. In contrast, the dynamic view utilized the language of motion and movement via slope. It was initially conjectured that this latter view would offer more conceptual complexity than the static view and that there ought to be evidence supporting a progressive mathematization from the static to dynamic view.

Following extensive data analysis, it turned out that these initial conjectures about the data needed significant modification. First of all, the static view turned out to be a powerful and conceptually developed idea contrary to expectations. Students came up with several sophisticated expressions of this view, including one involving a similar triangles argument. There was also no evidence for a direct, causal link between the static and dynamic views. One conclusion was that the static view needed to be revised and broken down into two new views, calculational and proportional. The calculational view would be one in which the student only focuses on the slope as a numerical value and the proportional view would be the one supported by such ideas as the similar triangles argument. The reason for the lack of an observed connection between the static and dynamic views might have to do with one view being stationary and focused on the present structure of slope and the other being fluid and focused on the predictive nature of slope. Like the Heisenburg Uncertainty principle from physics, this finding might suggest that one cannot know both what something is and where it is going simultaneously.

Another finding of this study was that there were two types of reasoning, additive and multiplicative, that characterized students' thinking about the use of the x(t) and y(t) equations for the SLSs. Additive reasoning was characterized by the students performing a comparative analysis of the exponents of the two SLS equations to determine which one had the greater force or pull for initial conditions not lying on a straight line. Multiplicative reasoning, on the other hand, was evident in students making explicit use of the ratio y(t)/x(t) in their reasoning about the long-term behavior of these non-straight line solutions. On the final exam, students were given a question that tested their ability to apply the ideas of additive and multiplicative reasoning to sketch the graphs of solutions not on a SLS. It was initially conjectured that students who used multiplicative reasoning would perform significantly better than those who did not. Another conjecture was that these two methods, additive and multiplicative, would function independently of each other.

Following a thorough review of this data on the use of the x(t) and y(t) equations, it turned out that these initial conjectures were incorrect. Exam results showed that students who used additive reasoning actually performed slightly better than those who used multiplicative reasoning. It was speculated that this was possibly due to difficulties with calculating limits of quotients of exponential functions. There was also data to refute the conjecture that these two methods were mutually exclusive. Several students who used one form of reasoning on an exam used the other one on the final. It seems that there are benefits and drawbacks of using either method to graph particular solutions in the phase plane. It is likely that students become aware of these limitations and use whatever method seems most helpful and computationally efficient at the time.

LIMITATIONS AND WEAKNESSES

The purpose of this section is to identify constraints and other factors that may have limited or weakened this study. Limitations and weaknesses are not seen as separate entities, so they will be discussed jointly. It is my hope that in future studies these issues can be addressed to make for stronger data collection and analysis.

One limitation of this study was the experience and level of expertise of the graduate student researchers. For many of us, this was our first major research project and there would be many mistakes involved in learning the practical and theoretical elements of this study. Fortunately, the guidance and expertise of Chadley on this type of research enabled us to get up to speed more quickly.

Another limitation was the lack of collected classwork from students during the days of the CTE. This made the video analysis from those days more challenging because it was difficult to discern exactly what students were doing and writing down in their groups. In future studies, it would be advisable to establish a more organized way of collecting student work so that there would be no missing data.

Although grounded theory analysis does not require the analysis of many students for successful research, it may have been helpful had there been a third cameraman responsible for another group of three students. This way, they could capture the activities of a group in the middle of the class and provide a perspective on the data collection missing from this study, which only had cameras at the front and back of the classroom. This cameraman could have even panned on two separate groups in the middle of the classroom, allowing for a more comprehensive collection of data and the possibility for more conceptual categories of data analysis.

Likewise, it would have been beneficial to conduct an analysis of more end-of-semester interviews. Of the twenty-one students interviewed, only five of them were used for this study. Transcribing and analyzing each interview is a time-consuming process and there simply was not enough time to be able to do a more comprehensive analysis.

Completing these analyses in the future, however, could prove to be beneficial to further the data analysis started in this thesis.

The java applet demonstration used during the April 13, 2005 class session could have been improved so that it did not stir up so much confusion among students. During this session, students were supposed to algebraically determine the slope of the SLSs to the system of ODEs modeling the spring-mass problem. Since the approximate location of the point was revealed on the applet and many students were aware that a line is determined by two points, they were confident they could find the equation of the SLS by just finding the slope of the line between the point given on the applet and the origin. After much debate over the authenticity of this point, the teacher had to tell the students that they could not assume that this point was on the SLS, thereby confusing them further because they wondered what the purpose of putting the coordinates of that point on the screen was in the first place. This problem could have been remedied in one of two ways. Either the teacher could have just given the students a vector field without any points on it and a laser pointer could have been used to simulate the location of a point in question or the crosshairs could have located a point without providing the coordinates. That way, students would have no way to be able to assume that the coordinates of the point on the SLS are known, thereby eliminating the confusion.

RECOMMENDATIONS FOR TEACHING AND FURTHER RESEARCH

There would be little purpose to this study if there were not significant consequences for teaching and further research that evolved from it. One of the central motivations for conducting this research was so that its conclusions could be of practical value to those seeking to understand more about how their students think and reason about systems of linear ODEs.

The findings on the static and dynamic views of slope could be of significant use to teachers and researchers. Teachers could construct activities that focus on the calculational, proportional, and dynamic views of slope. For instance, given a system of linear ODEs, students could be asked to first determine the slope of the SLSs. After finding the slope, they could answer a series of questions that probe their understanding of what the slope value means to them. This could include a question asking students whether they agree or disagree with the statement that the SLS is really made up of infinitely many individual vectors with constant slope, similar to the unitizing argument used by Mario.

Researchers, on the other hand, could perform further research on these two views of slope to determine if there is a connection among the three. There is no direct evidence in this study to support the existence of such a link but the data collection was not sufficient to rule it out. In fact, it still makes intuitive sense that such a link ought to exist, even if it is not in the form of a progressive link. It makes sense that students might need to make use of a dynamic view of slope in certain situations involving the ideas of motion and movement and to make use of the static view during formal calculations and when graphing SLSs in the phase plane. Furthermore, it seems that the definitions of “static” and “dynamic” may need to be modified further and perhaps broken down into more subcategories as was suggested in an earlier chapter.

The findings on additive and multiplicative reasoning might also be of interest to teachers and researchers. When teaching students how to graph solutions to systems of ODEs in the phase plane, it might be constructive to design activities that support the use of both views. As seen in the data analysis, students made use of both methods throughout the semester, depending on the nature of the problem they were given.

One point of curiosity for future research is why students who used additive reasoning on the final exam problem performed better than those who used multiplicative reasoning. An initial conjecture was that the use of explicit ratio involved more complicated calculations than the “force” argument. Of course, it could have just been that the design of that particular problem on the exam lent itself more easily to additive reasoning, but further research is necessary to find the reason for this occurrence.

It would also be constructive to analyze how students solve novel problems involving SLSs. During the end-of-semester interviews, students were asked to solve a problem in which there were infinitely many SLSs. Depending on whether they solved the problem with the slope-first method or the eigenvalue-first method, they either got the identity “m=m” or “x=x,” respectively. Due to time constraints, I was unable to incorporate this research into my study, but it is still a point of curiosity. This novel situation is reminiscent of what happens to students in intermediate algebra when they solve a system of two equations in two unknowns and arrive at a contradiction or identity. It would be interesting to examine the connections between how students resolve that difficulty in intermediate algebra and compare it to the way students deal with the conceptual difficulty in differential equations.

A final point of curiosity is whether there is any correlation between the level of academic performance of students and their preference for either the slope-first or eigenvalue-first method for solving systems of linear ODEs. Of course, the terms “high performance” and “low performance” would need to be clarified, but perhaps a good starting point is to divide the students up into two or three groups, corresponding to a subset of letter grades. Perhaps the results of this research could be used to determine if students’ use of a particular method may have contributed to their conceptual understanding of systems.

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APPENDIX

STUDENT, TEACHER, AND RESEARCHER QUOTATIONS

1) Teacher: If this were a SLS meaning that it came in this way [points to initial condition at (-1.5, 1.5)], physically speaking what would be going on? Let's just throw some ideas out here. We might have to revisit this in a little bit. Yeah.

Efrain: I think we said it's the point when the friction is the negative of your velocity such that it decreases your position and your speed decrease at a similar rate where um I couldn't think of a real good example of it but driving a car you let go of the gas you slowly slow down the car the engine cuts and you just slow, but if you go into a wall you abruptly stop because the friction takes over the…overtakes the speed of the car. That'd be much more abrupt. In this case it would be something where your velocity would be exactly equal to the negative friction or friction equals negative velocity.

Teacher: Ok, so there appears to be I'm asking you guys a slightly deeper question now not just where are we, but what are we doing? So ok we have this idea of something going on with friction. Ok. Something's going on with friction…and what I'd like let's see. Chris, maybe you can rephrase what Efrain just said maybe in your own words and add to it.

Carlos: Well he was saying that if uh the friction force is going to be the same as the force that's exerted on the mass then

Teacher: Ok.

Carlos: it would cause a straight line because um it would then give you equal forces. If you pull the force of the mass out to a point where um both of these forces are equal then it's going to slow back to the equilibrium point.

2) Timothy: How about when the rate…the two rate are equal? x and y are equal? dx and dy?

Larry: What would that do? So the rate of changes are equal.

Timothy: See the rate of change here. This here equal here equal. So set these equations [Timothy is referring to the dx/dt and dy/dt equations, respectively] equal to each other solve for…

Larry: Ah, what an idea there. Wow.

Timothy: Solve for x…Ok, let's try that.

Larry: Alright.

Timothy: So I say that we say that it's um if I put the two equations equal to each other meaning the rate of change in position is equal uh the rate of change of velocity so

Larry: Which would mean that it's decreasing.

Timothy: Proportionally…

Larry: Mhmm.

Timothy: Proportional to each other.

Teacher: Oh ok. So it's the idea of a proportion here…

Timothy: Proportion. Yeah, so if we have that and it look like a straight line.

Larry: Yeah, so we set it equal to each other and found that.

Teacher: Ok. Interesting.

3) Efrain: If you're going to zero and your velocity is decreasing why is its rate of change positive?

Timothy: So you say…

Larry: The y is actually the rate of change of position.

Timothy: You say the y?

Efrain: The velocity is the the velocity equation dy/dt is equal to -2x-3x so if your velocity is decreasing why is your equation for it positive?

Timothy: Well it decrease like a car. Like your example with the car. You release the acceleration the velocity will decrease but you're still gaining your displacement.

Efrain: I guess similarly if your position is increasing or nevermind nevermind that was fine. I was confused why the dy/dt was positive.

4) The assumptions I made changed slightly. Before our Wed. lecture [April 13, 2005 class] me and my group found and assumed there was a proportional relationship between dx/dt and dy/dt. We found a negative proportion of dx/dt to dy/dt and used this information to find our slope, which turned out to be 1 of 2 slopes (working through the problem in class).

My understanding of the slope of straight-line solutions before Wed.'s lecture was in relation to the direction, positive or negative, of dx/dt to dy/dt and their proportions. During Wed.'s lecture we looked at the slope of a straight line in relation to the positive general proportion of dx/dt (change in x) and dy/dt (change in y). The relationship was used as m=(dy/dx), the slope we were to find.

The teacher helped in elaborating this relationship a little further. Thinking back to Algebra and Geometry when you dealt with triangle proportions with a small triangle within a larger triangle. Legs of big triangle are y (straight line down from initial condition to x-axis), x (from origin to x value in initial cond.), hypotenuse is the straight line going from initial condition to the origin. Legs of small triangle are dy/dt (vector along the y leg starting at init cond.), dx/dt (line going from tail of dy/dt vector to the hypotenuse of big triangle), and hypotenuse (vector along big triangle hypotenuse starting at init. cond. and tail touching dx/dt leg). Where dx/dt of small triangle is proportional to x in the big triangle and dy/dt (small triangle) is proportional to y (big triangle). Set up the ratio and you get the relation of (dy/dx)=(y/x) a slope of a straight line.

5) Int: What I'd like you to do is explain why you set up the equation initially in this way with this ratio [m = -3x+4y/2x-y].

Timothy: What does the ratio mean?

Int: Yeah, what does the ratio mean?

Timothy: Ok. So first…

Int: How do you interpret this ratio?

Timothy: How do I interpret that ratio?

Int: Yeah.

Timothy: Ok, m represents the slopes…and this, the top part, is dy/dt, which is the change in the y-axis with respect to time…and then over this…dx/dt, which is the change in x-axis with respect to time, so you take the ratio of that, I will have a slope…or a changing

Int: Mhmm.

Timothy: of two axes…

6) Efrain: It says to see if the solution goes to the origin along a straight line then dx/dt should have a linear relation to dy/dt. So we should set dx equal to dy/dt to see if there's a relation like this. But we notice if you start at point P on that graph your velocity which is the dy/dt goes negative as it approaches the origin and your position even though it starts negative is decreasing positive as it goes to the origin. So we should set our uh our dx/dt equal to positive but our dy/dt is a negative. We set –dy/dt = dx/dt and we uh solve that put in the values that uh dx/dt and dy/dt has. You get y=-x. So that it is on a straight line with slope -1.

Teacher: Are there any questions for Efrain?

Roberto: I don't understand why you set dy/dt equal to dx/dt.

Teacher: Ok, that's a fair question. So uh then this was sort of an issue that we discussed at the end of class I think with Timothy's group or Roberto…I think Timothy's group about that underlying assumption. Could somebody maybe recall for us what that underlying assumption was we agreed upon that we weren't able to make?

Samantha: We agreed there was a SLS.

Teacher: Well we did agree, yeah we agreed there was a SLS, but what about the SLS? What about its slope?

Samantha: It would be proportional. That it's the velocity over displacement.

Teacher: Ok, there's some sort of proportion we set up. Yeah Bryce.

Bryce: Well I think by setting up the –dy/dt to the dx/dt, you're assuming that you have a slope of -1.

Teacher: Kathryn.

Kathryn: Ok, I was going to agree. The initial assumptions that you have a slope of -1 so that initial assumption you're just going to prove your initial assumption with your assumption. I don't know.

Teacher: So the issue here is the reasoning seems to be a little bit circular.

Kathryn: Yeah.

7) Efrain: Ah my reasoning for that is with the computer application you can't get an exact point. An x that's -1.46 and a y that's exactly 1.46. You can't accurately graph that exact point so every time you move it it's going to move a little infinitesimal point and it's never gonna be able to match up exactly. We assume that they want you to match up at that point because otherwise…of course you can't find a straight line that goes there because even with a slope that's gonna reach 0 on the y-axis first.

8) Mario: Okay, because the idea for proportionality makes sense. If we want a straight line solution um to solve for it because that would um, I have to go to linear algebra with those terms again. Okay, that would serve as a basis 'cause they have to be linear independent for a basis to occur. And, for that though, we need slopes that would find that. Umm, I guess a straight line solution makes sense to me because well you can pinpoint numbers. Okay, why did I pick this one dy/dt? Well, when we first talked about proportionality to solve for certain things, we had to find the slope 'cause it goes from a certain point all the way to zero and from there we figured out that those slopes that we figured out were basis, or bases, yes bases. Yeah, and then from there, because they were linearly independent, um, if you took linear algebra, you'd, I guess people would understand that um you can pinpoint any numbers on the R2 plane and we're only dealing with the R2 plane because we're doing the coordinate system, x and y values and extra variables. And since the phase portrait is only on a two value system, or you know, y and x, you can really use this so, I guess why it just makes sense to me.

9) Int: What is the relationship between this here: y = mx and dy/dx?

Mario: Okay, for a straight line solution to occur, um, y/x has to equal dy/dx. Because, if the, okay, if the rate of change was different than that of the original slope that means it curves off because it is not going in the same direction. It's just saying, oh, if it was less than, it would pull it to one side and if it is greater than it would be pulling off to the other side. It has to be equal in order for a constant line to exist. So, that is the relationship I see between the two. Um, what was the question?

Int: What is the relationship between y = mx and dy/dx?

Mario: What is the relationship? Um, Okay, well, y = mx, if you just change that around you get y = x, y/x= m, which is the slope and since dy/dx one is the derivative of the top and one is the derivative of the bottom, it's the roc, for one of each.

Int: Okay.

Mario: So, in a sense, um, the roc is like the future of y/x, so in a sense that because, since it's future, um, we can predict what the next m could be I guess. That's I see a similar relationship between each other or one predicts each other. Like the y/x = m could predict dy/dx, but then dy/dx would predict the next y/x. or = mx. So, they intertwine with each other.

Int: Excellent.

10) Int: So what's the relationship between this equation, right here. (Int points to y = mx) and dy/dx? For you.

Li*:* For me, it's almost like two halves of one coin. I mean, it's saying that, it's saying that there is a change along this, you know, slope, this m that we don't know yet of x and y and it just depends on what that constant m is. What the slope is and that'll determine the dependencies.

Int: So that's the relationship to this (Int points at dy/dx) and that (Int points at y=mx)?

Li: Yeah. Yeah.

11) Mia: For the exercise in class my group really did not understand the task as "proportional" reasoning. We were simply using past techniques as a way to solve for the slope. From the bees and flowers example (I think) the class determined a way to find the slope would be (dy/dt)/(dx/dt). The reasoning for this was rise (dy/dt) over run (dx/dt). So for the new task we used this technique. We used rounded values of 1.5 and -1.5 for y and x. That would give the solution of -1 slope. After today's class I have learned it is incorrect to assume x and y values and so we have to take what is known:
y=mx + 0

(dy/dt)/(dx/dt) and using substitution of the definition of a slope for y values in the determined (dy/dx) value.

12) Kathryn: I believe it can be helpful because the slope of a line is simply how much y changes with respect to a change in x. If you look at a change in y of 1 (for example), the x value will change a finite amount. If the equation is a slope of a straight line the change in x will always be the same if the change in y you take is 1. For example, if you started at point (2,3) and the slope was 1/2, then the next point would be (3,5), exactly 1 unit in y and 2 units in x. If you started on the same straight line, then the next point would be (4,7). The proportion between each post is always 1/2--the slope. So if you have a system of differential equations, and you can find out what is the proportion of their rate of changes, you can find the slope of straight line solutions.

13) Bryce: For a straight line solution through the origin - y must be proportional to x. That is to say if delta(y)/delta(x) = (y1 - 0)/(x1 - 0) = y1/x1 = m --> since the slope is a constant any point (x,y) on the line must have the same ratio m. So y = m*x, which really should be no big surprise, but I guess the slope is not usually thought of as a proportionality constant.

14) Hopi: Well before Wednesday's class I made the assumption that you could use a proportion to find the slope of the straight line solutions. So I did use dy/dt over dx/dt to create a proportion to help me find a slope. But then I wasn't sure what assumptions I could make and could not quite yet see how I could find the slope algebraically so I didn't have to make any uncertain assumptions. Wednesday's class really clarified a lot for me. By substituting y=mx into the proportion, then I could find the slope m=-2 and m=-1 that would give me my straight line solutions. I knew initially that I would have to get a negative slope since the point on the graph was being pulled to the center, but I didn't know if I could assume that y=-x. I also wasn't sure if the point was on the straight line solution because we had not yet proved this before Wednesday. Now I know that there can be more than one straight line solution and I would never been able to see this if I hadn't proved this algebraically or had gone by the very popular assumption that y=-x.

15) Li: My first instinct was to say that it was linear in time – that it approach that throughout time it was constant in going to zero…

Larry: Could you…well no…

Li: But I’ve heard a couple people saying that it’s going to be negative exponential so—

Larry: Really?

Li: Yeah because dx x as x changes with y in time it’s gonna to be an exponential.

Larry: Yeah.

16) Li: Well let’s think about this. If it’s a really really really perfect system you have a block on a spring, I could see it being linear.

Larry: Yeah.

Li: But if you're taking into account everything else like when a door closes it goes fast at first

Larry: Then slows.

Li: Then it slows. I don't know. I'm torn between them. I guess it depends on the system.

17) Teacher: Ok. Possibility 3! So Danny says like a decreasing exponential because an exponential wouldn't hit, right? So if we just say now it's exponential decay. So here we have it doesn't…hit zero. Alright, now, if we look on page 2.5 your task is to figure out what the equations are for x and y(t). Here's the thing. If you figure out what the equations are, you'll know the answer…which one of this it really is. So all three of these right now are totally reasonable possibilities for what the 3-D graph looks like. I would say you don't have any way to know which one of these is correct, but we can figure out which one is correct if we know what the equations are for x(t) and y(t).

18) Efrain: Could we…do you think we could put this into here? [to Mario]

Mario: What?

Carly: And derive it that way?

Efrain: Could you substitute in you think?

Mario: How?

Carly: See our major…but I don't know. Our major problem is that in order to get these so far we know that we need to derive these equations. Now there's another way—

Efrain: In terms of only y…in terms of only y…or x.

Carly: Right. So we would need to either get this in terms of x or t and this in terms of only y or t.

19) Timothy: I'm kinda lost.

Opie: Yeah, what are you trying to pursue?

Timothy: Well I'm trying to solve x uh y(t) and uh so first I try to uh no substitution. So integrate that side. Gonna have (y^2)/2 + C.

Opie: Yeah, where did you get that equation? (y^2)/2?

Timothy: This from here [points at dx/dt=y equation]. I integrated. I think I did it wrong cause uh yeah.

Li: That's the thing. You can't integrate with respect to two different variables x and y.

20) Mario: dx/dt=y and the initial conditions fall on the slope of y=-2x it's safe to assume that we can plug in -2x into or equal to dx over dt. Another reason is that x is position, y is velocity so the derivative of position is velocity so dx equals y, so, and we have a slope that [inaudible].

Teacher: So you get a lot of reasons here. Let me pull out one…is the first one that he said. We already figured out algebraically that if you start off with a vector that has this slope over here -2 then it keeps going with the slope it keeps going with the slope it keeps going with the slope. All the vectors along here have the slope of -2. All of these vectors which make up the solution. Solutions are then followed out by the vectors. Those all those vectors have a slope of -2. Those vectors lie along y=-2x. So that provides me the warrant that provides me the license to say for that initial condition. Now that's not true if my initial condition's over here. That's not true if my initial condition's over here. It's only true if this initial condition is along this line y=-2x. Then algebraically we've already proven that all the vectors stay on that line. So therefore I can substitute y for -2x.

21) Teacher: So the possibilities are the solution is either going to go towards zero and exponential-like, but it hits zero, or exponential decay, but it doesn't hit zero, or it's a straight line. Well, we've already said it's not a straight-line. We figured out x(t) is exponential. So does it actually equal zero?

Maya: No.

Teacher: Well, we know exponential functions decay toward zero but not ever equal zero in any finite amount of time. When we answer the question about what the 3-D graph should look like, it's not possibility #1. It's not linear. It's not possibility #2. It doesn't actually hit zero. It decays exponentially toward zero.

22) Efrain: I would disagree with that cause we proved that the slope of that line was negative was y=-2x and so why would they get closer if they're following the same slope the whole way down? What would cause them to…what would cause the slope to change such that one would catch up with the other or if I guess that's what's happening?

Teacher: Ok, we have several hands I want to respond. Timothy, why don't you go ahead and respond.

Timothy: Uh because when we have a graph of a 3-D we have a exponential decay so as it moves down to zero it's gonna go faster. Decay faster approaching zero. When it's far from zero, it's gonna go faster. When it's close to zero, the two points are very close to each other.

Teacher: Is Timothy saying they're gonna move faster…they're the points…are they gonna move faster here [further away from origin] or are they gonna move faster when they're in here [closer to origin]? What's Timothy saying?

Timothy: Faster out.

Teacher: Faster out here. Roberto can you can you build on Timothy's idea and show us what the 3-D graphs would look like.

Roberto: It's kinda like thisish. It's going off into space but these two kinda arc off towards the center.

Teacher: Ok, everybody see how he's arcing?

Mario: I don't think it arcs that way. I think it arcs more like it goes down like you know exponential graph.

Larry: Cause t comes out of the board and you traveling along the positive [inaudible].

Teacher: it's coming out of the board, right? So Mario, you want to come up and show us what you think the 3-D graphs look like?

Mario: I'm thinking there's like there you know like what Timothy said that [inaudible].

Teacher: Timothy?

Mario: Like he said something like exponential decay and that's what it was for x(t) and y(t) so I'm thinking that from these points that they curve down like that towards um zero but not really touching.

23) Teacher: So we have that x(t) is -2e^-2t and y(t) is 4e^-2t. Think about this e^-2t as your speed controller. So here you are a time 0 and t=0. That's just one. So you start off at (-2,4). Now as time goes on this is your speed controller and when time is 1, 2, 3. What happens as time gets up to like 10, 20, 30? What happens to this number?

Mario: It gets smaller and smaller.

Teacher: Smaller and smaller. So how do you interpret that in terms of speed?

Mario: Slows it down.

Teacher: Slows it down. So it's that would be a way to justify or think about the changing speed in the phase plane.

24) Int : Can you, umm, when you're done…explain how you came up with those shapes?

Timothy: Alright.

Int: What they mean.

Timothy: So at the beginning, I say this is gonna dominate the eigenvalues and go back to here…if I take the limits of this…as time approach to infinity, this term will uh not able to compete with this term…because exponent e^5t is a multiple of 5…is significantly bigger, so this term will be dominating…therefore any initial initial condition…

Int: Mhmm.

Timothy: is going in the long run, the component of this one will win out versus this component…
Int: Hmm.

Timothy: so that's what the shape will look like. Alright, and I think that's my test.

25) Li: Let's see. Okay, so at t equals minus infinity, let's see you have. What does it tend to at minus infinity? Let's see (Inaud.) e to the minus infinity plus e to the -5 infinity. Okay. So obviously, the stronger term at minus infinity is going to be 5. It's just going to go closer. So the stronger one at t equals minus infinity is y equal -3x.

Int: What do you, what do you mean stronger? I don't see a person here or…

Li: It has…

Int: Or anything of that nature.

Li: It's like that whole force thing we talked about in class. You know how we talked about in force like at this term (Li points to) is going to go to zero. Like, at negative infinity, it's going to be closer. This line that corresponds to e to the -5t is going to pull on the solution stronger than this one. Cause e to the -5 times infinity is a bigger number than e to the minus infinity.

26) Mario: Um, we can also do this by limits, so let's say, what um, as t goes to infinity, the limit of that at, e^t + e ^ 5t with e^t +, no, no, no – 3 cause we're doing it y/x, we want to figure out the slope of it. A slope of the solutions as t goes to infinity, so it's going to be y/x, so instead of e^(t+5t), it's um e^t-3, e^5t/e^t+e^5t. Now what we can do is factor out a e^5t for both of them, so we get e^5t, multiplied by, which would cancel. So, we would get e^-4t-3+1 as t goes to infinity, the e^4t's would go to zero, so it would be 0-3 and 0+1, so the slope of it as t goes to infinity would be um -3.Though, what it is saying is that as t goes to infinity, it branches off and it's going to follow a slope similar to the one we have for y = -3x because you know, it's going to follow that. It's not going to be exactly like that, but it's going to follow that or try to be parallel as [inaudible] run parallel to it.

27) Anita: Ok, so now I have these two lines I can take a limit of y/x for t going to infinity and for t going to negative infinity just to see where the graphs will be approaching for both time periods and it is, yes, yeah of course I have to go through all that because umm the solution x(t), y(t) For the first one the solutions are, equals e^5t and it was –3x, so x is 1 and t is negative, I mean, y is –3, so limit of, we need another one. x(t), y(t) because we need to find [particular] solutions.

28) Carly: Now I do the same thing as the limit goes to negative infinity and I'm doing this because the limit is going to be telling me what my slope is going to be looking like which uh for the different graphs on the phase portrait which I already um well that doesn't make sense. Ok…well let me do the other one and then I can make…I can come to some conclusions but from my initial I said that it was going to be tending…oh no, I'm [inaudible]. For my initial I said it was going to be tending toward equals 5t e^5t which will look like a -3 slope umm [inaudible] ok.

29) Carly: Ok – I was right then cause as it goes toward negative infinity it's going to look like the negative 3 slope um pulling towards the negative 3, and as it goes to positive infinity

Int: Mhmm.

Carly: It's going to have a…it's going to have a limiting slope of 1and as t goes to negative infinity it's got a limiting slope of negative 3.

ABSTRACT OF THE THESIS

Students' Reinvention of Straight-Line Solutions to Systems of Linear Ordinary Differential Equations
by
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This paper analyzes how students who work in an inquiry-oriented classroom think and develop solution methods for analytic solutions to systems of two linear ordinary differential equations (ODEs). In particular, we seek to understand how students construct and interpret straight-line solutions (SLSs). SLSs are significant mathematical ideas because they serve as the basic building blocks for all other solutions. In particular, SLSs are eigensolutions that span the solution space. This report will also detail the role of proportional reasoning in the process of reinventing SLSs. Our theoretical orientation comes from a socio-constructivist view in which individual math activity is related to emerging classroom meanings and practices. Analysis of student work and justifications suggests that their mathematical understanding of SLSs is a dynamic and evolving entity that matures through classroom discussion and activity. Lastly, this study examines implications for teaching and for the revision of instructional materials.