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Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions

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Abstract

An enduring challenge in mathematics education is to create learning environments in which students generate, refine, and extend their intuitive and informal ways of reasoning to more sophisticated and formal ways of reasoning. Pressing concerns for research, therefore, are to detail students’ progressively sophisticated ways of reasoning and instructional design heuristics that can facilitate this process. In this article we analyze the case of student reasoning with analytic expressions as they reinvent solutions to systems of two differential equations. The significance of this work is twofold: it includes an elaboration of the Realistic Mathematics Education instructional design heuristic of emergent models to the undergraduate setting in which symbolic expressions play a prominent role, and it offers teachers insight into student thinking by highlighting qualitatively different ways that students reason proportionally in relation to this instructional design heuristic.

Keywords: Modeling; Undergraduate mathematics; Realistic mathematics education; Student thinking; Proportional reasoning

An enduring challenge in K-12 mathematics education is to create learning environments in which students generate, refine, and extend their intuitive and informal thinking to more sophisticated and formal ways of reasoning. In the last decade this challenge has been increasingly felt at the undergraduate level as well for a variety of reasons, including dissatisfaction with student achievement, new pedagogical challenges as colleges accept larger and more diverse groups of students, and a need to increase the number of mathematics and science majors (Holton, 2001). Part of the response to these challenges is to develop new curricular approaches based on contemporary theories of instructional design in which students reinvent mathematical ideas for themselves in conjunction with classmates and their teacher. A pressing concern for research and practice, therefore, is to characterize the processes by which this can happen. Responses to this concern, such as the one in this article, are useful to inform principles of instructional design and to offer teachers insights into student reasoning.

In this article we characterize the process by which students reinvent solutions to systems of two linear, homogeneous differential equations with constant coefficients. The data comes from a research driven approach, referred to as the inquiry-oriented differential equations (IO-DE) project (Rasmussen, Kwon, Allen, Marrongelle, & Burtch, 2006). The IO-DE project adapts to the undergraduate level the instructional design theory of Realistic Mathematics Education (RME) as a way to engage students in the reinvention of important mathematical ideas in differential equations.

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The primary contribution of this work is to further develop the instructional design theory of RME by advancing the design heuristic of emergent models. In general, the emergent model heuristic offers a transition from model-of to model-for in which learners create new mathematical realities (Gravemeijer, 1999; Rasmussen, Zandieh, King, & Teppo, 2005; Zandieh & Rasmussen, 2007). As others have noted (Gravemeijer, 1999; Zandieh & Rasmussen, 2007) the shift from model-of to model-for and the resulting new mathematical reality is compatible with Sfard’s (1991) process of reification. In prior research, most of which has been done at the K-12 level, the model-of phase has not involved analytic expressions, but rather concrete manipulatives, graphs, or diagrams. Only later in the model-for phase do students work with formal, symbolic expressions.

In contrast, our work at the university level has found that analytic expressions can function as tools for reasoning in all phases of the model-of/model-for transition. As far as we were able to ascertain, this is the only example of analytic expressions underpinning the entire model-of/model-for transition. Because students’ mathematical work in undergraduate courses is often driven by analytic expressions, the case of differential equations reported here offers evidence that RME can serve as a useful instructional design theory for mathematics instruction at the undergraduate level. Research in abstract algebra (Larsen, 2004) offers additional evidence to support this assertion.

At the more pragmatic level, the analysis offers teachers a form of pedagogical content knowledge (Shulman, 1986) by detailing qualitatively different ways that students reason proportionally in relation to the model-of/model-for transition. Explicit awareness of student reasoning and knowledge of how these ways of reasoning are related to each other as well as how they differ is useful knowledge for teachers who desire to foster their students’ mathematical power (Rasmussen & Marrongelle, 2006).

1. Mathematical background

Linear systems of differential equations typically arise in the physical and natural sciences as a way to describe two or more simultaneous rates of change. Such systems can be formed as a way to more easily analyze solutions to higher-order single differential equations by reducing the order of the derivatives. For example, consider the second-order differential equation \( \frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0 \), which can be rewritten as a system as follows: set \( \frac{dx}{dt} = y \) and hence \( \frac{dy}{dt} = -6x - 5y \). The second-order equation can then be rewritten as the linear first-order system

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -6x - 5y
\end{align*}
\]

This method of order reduction, and subsequent analysis of the resulting system, is useful for ascertaining information about the behavior and structure of solutions not readily accessible from the single higher-order algorithm.

Systems of linear and nonlinear differential equations can also arise as a way to depict physical and biological situations without appealing to higher-order reduction methods. For instance, consider the system of nonlinear equations

\[
\begin{align*}
\frac{dR}{dt} &= 3R - 1.4RF \\
\frac{dF}{dt} &= -F + 0.8RF
\end{align*}
\]

where \( R \) represents the changing number of rabbits over time and \( F \) represents the changing number of foxes over time. This is an example of a predator–prey system that reflects competition for resources between two species, and is not derived from the method reduction of order, but rather from theoretical assumptions and scientific observations.

Eq. (1) represents the general type of linear system that students worked with in this research setting.

\[
\begin{align*}
\frac{dx}{dt} &= ax + by \\
\frac{dy}{dt} &= cx + dy
\end{align*}
\]  

(1)

Solutions to such systems are typically graphed in the \( x–y \) plane, referred to as the phase plane. The phase plane is a two-dimensional cross-section of three-dimensional \( x–y–t \) space, often shown with accompanying tangent vectors.
whose slopes are determined by the differential equations. A key idea for determining analytic solutions in the phase plane is that of eigensolutions, also referred to as straight-line solutions (SLSs) in the case when the eigenvalues are real. SLSs are significant mathematical ideas because they serve as the basic building blocks for all other solutions.

The term “SLS” comes from the fact that when such solutions are graphed in the phase plane, they lie along a straight-line of tangent vectors. For example, the phase plane (with tangent vectors) in Fig. 1 shows two different SLSs to the system of differential equations

\[
\frac{dx}{dt} = 2x + 2y \\
\frac{dy}{dt} = x + 3y
\]

The \(x(t)\) and \(y(t)\) equations for these particular SLSs are \(x_1(t) = e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) and \(x_2(t) = e^{t} \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\). At time \(t = 0\), \(x_1(t) = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix}\) begins at the point \((1, 1)\) and the solution is shown for positive values of time only. Similarly, \(x_2(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}\) begins at the point \((1, -1/2)\). How one can determine these solutions is detailed later in this section.

In the analysis reported here we deal only with the case when the eigenvalues are real and distinct. The approach students developed generalizes algebraically to the case when the eigenvalues are complex.

From an expert’s point of view, SLSs for a system of the form (1) are completely determined by the eigenvalues and eigenvectors for the corresponding 2 \(\times\) 2 matrix. Typical approaches for finding the general solution employ techniques from linear algebra in which students are first taught to find the eigenvalues, then to find the corresponding eigenvectors, and then to form the analytic solution. An example of this traditional solution method is given in the Appendix. Although students can often solve the characteristic equation to find eigenvalues and eigenvectors, the meanings of these mathematical ideas tend not to be well understood by students (Dorier, Robert, Robinet, & Rogalski, 2000; Rasmussen, 2001).

Rasmussen and Keynes (2003) describe an alternative instructional approach that is more consistent with students’ mathematical thinking. They call this approach the “eigenvector first approach” or “slope first approach” because it contrasts to methods of instruction in which the computation of eigenvalues precedes the computation of eigenvectors. In earlier design research efforts, Rasmussen and colleagues followed the conventional eigenvalue first approach. After
listening carefully to student-generated methods, however, it was discovered that focusing on eigenvectors first, which extends students’ strong mathematical and intuitive understanding of slope, was a more natural starting place for students.

In the “slope first approach” students develop their own method for locating lines of eigenvectors and corresponding solutions to systems of two first-order linear differential equations. The typical algebraic method that students invent as a means to determine the slope of a straight-line of vectors (such as those shown in Fig. 1 that lie along the lines $y=x$ and $y=−1/2x$) capitalizes on their knowledge that a line going through the origin is of the form $y=mx$ and that the slope of the vectors is the ratio of the two rate of change equations given in (1). Thus, $m=y/x$ and $m=(dy/dt)/(dx/dt)=cx+dylax+by$. Equating the two expressions for the slope yields $y/x=cx+dylax+by$. Replacing $y$ with $mx$ and simplifying yields a quadratic equation in $m$. Thus, depending on the discriminant, there are two, one, or no real values for the slope. Students initially work with systems in which there are two distinct lines of eigenvectors and later generalize to the other cases. Once the slope values are obtained, students typically substitute $y=mx$ into the differential equations (1) and then use separation of variables to find the general $x(t)$ and $y(t)$ equations for any solution that lies along the given straight-line. To find a specific SLS one need only to choose an initial condition along the straight-line. With two distinct SLSs this yields two linearly independent SLSs.

Once students have obtained two linearly independent SLSs, the principle of superposition is applied to create the general solution to the system of differential equations. For example, if a system of differential equations has two SLSs

\[
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix} = e^{t} \begin{pmatrix}
  1 \\
  1
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix} = e^t \begin{pmatrix}
  1 \\
  −1/2
\end{pmatrix},
\]

which lie along the lines $y=x$ and $y=−1/2x$, respectively, then all solutions are given by a linear combination of the two SLSs. In this example, the general solution is

\[
\begin{pmatrix}
  x(t) \\
  y(t)
\end{pmatrix} = c_1 e^{t} \begin{pmatrix}
  1 \\
  1
\end{pmatrix} + c_2 e^{t} \begin{pmatrix}
  1 \\
  −1/2
\end{pmatrix}.
\]

The general solution is thus a product of students’ own inventive activity.

2. Theoretical background

Traditional instruction at the undergraduate level tends not to encourage students to create their own strategies and techniques for solving problems. Over the past decade, however, mathematicians and mathematics educators have been exploring approaches that invite learners to build their own ideas and ways of presenting these ideas (e.g., Speiser & Walter, 1994; Zandieh, Larsen, & Nunley, in press). In our work we have found the instructional design theory of RME to be useful for these achieving such goals with learners. Indeed, RME is geared toward enabling students to invent their own methods of reasoning and solution strategies, leading to a stronger conceptual understanding. In fact, research by Kwon, Rasmussen, and Allen (2005) and Rasmussen et al. (2006) indicates that IO-DE students develop stronger conceptual understanding without loss of procedural competency, as compared to comparable students in conventional approaches.

The basic adage put forth by Freudenthal (1991), whose ideas inspired the development of RME, is that mathematics is essentially a human activity of organizing subject matter at increasingly more sophisticated levels. Mathematics as an existing collection of concepts and skills is secondary. Thus, from this perspective, students should learn mathematics through their own constructive activity. In RME-inspired curricula, students are provided opportunities to reinvent important ideas and methods for solving problems. Their learning is grounded in experientially real situations, which ultimately leads to the development of formal mathematics. What is experientially real for students depends entirely on their background and experience. For example, in systems of differential equations the experientially real starting point includes the slope of vectors (which also function as indicators of solutions via their rate of change), together with students’ underlying imagery of oscillatory motion for a spring-mass scenario.

In general, models are defined to be student-generated ways of interpreting and organizing their mathematical activity, where activity refers to both mental activity and activity with graphs, equations, etc. (Zandieh & Rasmussen, 2007). Such models are “emergent” in the sense that various ways of creating and using tools, graphs, analytic expressions, etc. emerge together with increasingly sophisticated ways of reasoning. In the case of systems of differential equations, as students gain experience with vectors and vector fields as manifestations of rate, their organizing activity with the differential equations function as a model-of their relevant work with vectors and the observed slope of SLSs.
The modeling process does not, however, stop at the model-of phase. The intention is that in subsequent instructional sequences, students’ mathematical activity changes from one of creating situation specific solutions to a model-for more formal mathematical reasoning. This global transition is referred to the model-of/model-for transition (Gravemeijer, Bowers, & Stephan, 2003). In the particular case of systems of differential equations, students’ work with analytic solutions for SLSs subsequently facilitates reinvention of increasingly more formal mathematics, including the general solution via the principle of superposition and as a means for justifying the structure and shape of all solution graphs in the phase plane. Thus, for students, models and modeling serve the function of creating a new mathematical reality. In particular, this new reality is the entire space and structure of solutions graphed in the phase plane for any system of linear differential equations of the form given in (1).

The model-of/model-for transition can be further detailed by specifying four layers for activity: situational, referential, general, and formal (Gravemeijer, 1999). Situational activity refers to acting in a particular task setting that is experientially real for students. Referential activity involves models-of that refers to activity in the original task setting. General activity involves models-for that facilitate a focus on interpretations and solutions independent of situation specific imagery. Finally, Formal activity involves reasoning with conventional symbolism, which is no longer dependent on the support of models-for mathematical activity. In the analysis of student reasoning section we use these four layers of activity to structure the discussion.

The following is an example adapted from Gravemeijer, Cobb, Bowers, and Whitenack (2000) of these four layers of activity from a class of first graders learning single digit arithmetic. Note that in this example, analytic expressions figure prominently only in general and formal activity.

- Situational activity—students act out the situation of loading and unloading passengers on a double-decker bus to determine the total number of passengers.
- Referential activity—students slide beads on an arithmetic rack to indicate persons getting on and off each of the two bus levels and to calculate how many people are on the bus. Students’ organizing activity with the beads functions as a model-of the relevant physical and mental activity in the bus setting.
- General activity—students’ organizing activity with the arithmetic rack functions as a model-for solving single digit addition and subtraction problems in ways that do not refer to the double-decker bus setting.
- Formal activity—students reason about numeric relations in ways that reflect new place value understandings and consequently use conventional notation without having to interpret or unpack the meaning of those symbols in terms of the arithmetic rack.

In our review of the literature on the emergent model heuristic, we found no instances in which analytic expressions underpin the entire model-of/model-for transition. Instead, the model-of phase usually includes concrete tools, graphs, or diagrams, and only later do analytic expressions function prominently in the model-for phase (e.g., Bakker, 2004; Doorman, 2005; Gravemeijer, 1994, 1999; Gravemeijer et al., 2003; Gravemeijer et al., 2000; Gravemeijer & Doorman, 1999), a finding that is perhaps not surprising given that most of this RME-inspired work has been done at the K-12 level. Because our work is at the undergraduate level, we have found that student organizing activity with things that are more symbolic in nature, such as analytic expressions for rate of change, the $x(t)$ and $y(t)$ equations for the SLSs, or even definitions for that matter (Zandieh & Rasmussen, 2007), can underpin the entire model-of/model-for transition.

As a second example of the emergent model heuristic at the elementary school level, consider Gravemeijer’s (1999) description of an instructional sequence in which students’ organizing activity with a ruler initially functions as a model-of iterating measurement units, evolves into a number line, which then serves as a model-for reasoning about mental computation strategies with numbers up to 100. The ruler initially serves as a tool for measurement; its manifestation dynamically transforms as students progress in their measuring activities. Symbolizations on the number line thereby function as a basis for reasoning mathematically about more complicated ideas and problems involving number relations. New solution strategies are formed by students through guided experimentation and can then be used to further investigate more complicated problems. Conceptually, how student progress in their thinking about number constitutes the cognitive aspects of the ruler-to-number line transition. In our work with systems of differential equations, how students progress in their thinking about ratios and proportions serves as the conceptual foundation for students’ reinvention of SLSs.

Students reason proportionally in many different mathematical settings from multiplication to calculus to solve proportion-specific problems and to reason about other mathematical ideas (Carpenter, Fennema, & Romberg, 1993;
Harel & Confrey, 1994; Kenney, Lindquist, & Heffernan, 2002; Lamon, 1994, 1999; Behr, Harel, Post, & Lesh, 1992). Our work in differential equations has found that, given the opportunity, students reason proportionally in diverse ways to refine, interpret and extend their own methods for finding analytic solutions.

For the purposes of this paper we draw on the work of Cai and Sun (2002, p. 195), who characterize proportional reasoning as involving “a sense of covariation, multiple comparisons, and the ability to mentally store and process several pieces of information.” Specifically related to SLSs, reasoning about slope can be represented symbolically by the mathematical equation \( m = \frac{y}{x} \). Conceptualizing relationships in a quantitative manner such as this as well as comparing ratios are common manifestations of proportional reasoning (Cai & Sun, 2002; Thompson, 1994; Thompson & Thompson, 1994). Other manifestations of proportional reasoning include forming ratios, working with pairs of equivalent ratios, and generating a class of equivalent ratios (Lobato & Ellis, in press). Certainly the proportional reasoning literature is vast, and a complete review of this literature is beyond the scope of this paper. Because the primary focus of this paper is on extending the RME heuristic of emergent models, we leave an analysis of how these differential equations students’ reasoning fits within this broader literature on proportional reasoning for a future paper.

3. Methodology

The overall research design of this study follows that of the classroom teaching experiment, as described by Cobb (2000). The emergent perspective (Cobb & Yackel, 1996), which posits learning as both a process of active individual construction and a process of enculturation into a community of mathematical practice, provides the theoretical basis for our work. Specifically, students’ participation in evolving classroom practices are reflexively related to their ways of engaging and using particular ideas. To say that the relationship between the local social world of the classroom and the mathematical activities of students is reflexive means that neither exists independent of the other—they mutually support and constrain each other. This dynamic view of learning, which is highly compatible with an RME perspective on instructional design, is also supported by Steffe and Thompson’s (2000) contention that mathematics is a “living subject” and not a pre-determined body of knowledge to be imported by learners.

This study was part of an ongoing 8-week differential equations classroom teaching experiment conducted at a large, public university with a diverse student population. A primary goal of the course was for students to develop a deep conceptual understanding of the fundamental concepts and methods for analyzing differential equations. A second goal was for students to advance in mathematical sophistication through mathematical investigation and argumentation, through the creation of personally meaningful solutions to problems, and by expanding their ways of communicating mathematical thinking and activity to others, both verbally and in writing. The course focused on the formulation of differential equations, methods for analyzing them, and the interpretation of their solutions from a graphical, analytical, and numerical perspective.

Each class session was 50 min in duration and the class met three times a week. The majority of the 37 students were mathematics and science majors. Approximately half of the students had taken or were concurrently enrolled in linear algebra. The text for the course was a sequence of challenging problems in which learners, with the proactive involvement of the teacher, built essential ideas and concepts from the bottom up (Rasmussen, 2002). For example, in first-order differential equations students essentially reinvented Euler’s method and the concept of bifurcation diagrams. Thus, when the unit on systems began students were used to exploring and building mathematically sophisticated ideas for themselves.

The classroom data used in this study came from five consecutive days of teaching, which were equally split between both authors. Typical class sessions began with an introduction to a new problem or a continuation of the activities from the previous day. The rest of the class consisted of cycles of small group work and whole class discussion. One of the responsibilities of each student was, for an assigned class session, to summarize the sequence of events, the arguments that were put forth, and the concepts and methods that were developed. These summaries were reviewed by the teacher and posted on the course website for all students to access.

There were two video-cameras set up in the classroom for this study. One was positioned in the front of the class and the other was located in the back of the class. During group discussion, the cameras were focused on two groups in particular, one in the front and one in the back. Following each class session, students’ notes were taken and photocopied for further analysis. We also retained copies of all students’ homework, discussion board postings, and examinations. We conducted semi-structured interviews with 21 students at the end of the semester. Interviews were videorecorded and transcribed for subsequent analysis.
Based on earlier conjectures of Rasmussen and Keynes (2003), as well as our own experience as teacher–researchers in the course, our initial goal for the analysis was to uncover the diversity of ways in which students were reasoning with ratios and proportions as they reinvented solutions to systems of linear differential equations. We therefore engaged in what Strauss and Corbin (1990) refer to as open coding, which is the process of selecting and naming categories from the analysis of data. Specifically, we began the analysis for diversity of student reasoning with ratios and proportions by first examining five end-of-semester interviews with what we felt were particularly articulate and reflective students (two students from the front group, two students from the back group, and one particularly vocal student not in either group). This analysis provided us with a snapshot of student reasoning at the end of the learning process.

We then examined the transcripts of classroom data from the five relevant days of the classroom teaching experiment to better understand the histories of these learners in terms of how their ideas developed in the context of the classroom. These analyses resulted in identification of the following four themes characterizing the diversity of students’ proportional reasoning: why it made sense to set up the ratio $y/x = dy/dx$; the relationship between $y = mx$ and $dy/dx$; the role that proportional reasoning played in the long-term behavior of particular solutions; and the behavior of SLSs. We then triangulated this analysis using the constant comparison method (Glaser & Strauss, 1967) by examining discussion board posts, exam data, and the remaining 16 students who participated in the end of the semester interview.

We then engaged in the process of making explicit connections between categories and sub-categories. This step is what Strauss and Corbin (1990) refer to as axial coding. The aim of this step was to put our analysis together in new ways. Specifically, we came to see our analysis as a paradigm case of how analytic expressions could function as an integral part of the entire model-of/model-for sequence. As Strauss and Corbin (1990) argue, a researcher’s ability to see an analysis in new ways stems largely from his or her theoretical sensitivity. Sources of theoretical sensitivity include the research literature, professional experience, and personal experience. Certainly our experiences with the instructional design theory of RME were profoundly influential in the ultimate framing in terms of the emergent model heuristic. It is also important to point out that prior to this analysis, there was not a clearly articulated trajectory for the reinvention of analytic solutions to systems of linear differential equations in terms of the emergent model heuristic. As such, we expect this analysis to be pragmatically beneficial in future work with undergraduate differential equations instructors, as well as theoretically useful in extending the theory of RME to the undergraduate setting.

4. Analysis of student reasoning

In this section we analyze the role of analytic expressions in the emergent modeling process and students’ qualitatively different ways of reasoning with quantities during the reinvention of solutions to linear systems of two differential equations. We begin by describing the instructional setting in which students first recognize the existence of SLSs. This provides essential background that will be useful in understanding the model-of/model-for transition. We then exemplify the model-of/model-for transition, using the four levels of activity to frame the analysis of students’ mathematical activity. The first two levels, situational and referential activity, characterize the model-of phase, while the latter two levels, general and formal activity, characterize the model-for phase. We caution to note that these different levels offer a journey through students’ mathematical thinking, without imposing a strict hierarchy.

4.1. Instructional setting and the recognition of SLSs

As background for the subsequent analysis we begin with the task in which the teacher introduced the spring-mass scenario shown in Fig. 2 and invited students to create what they imagined to be all possible velocity vs. position graphs depicting the motion of the mass. The position vs. velocity plane is, in this case, the phase plane. The second-order differential equation that describes this situation was not yet developed. Thus, it was expected that students’ predictions would be drawn largely from their experiences and concept images (Vinner & Dreyfus, 1989) without the aid of technology or other outside resources.

Students’ predictions included the frictionless case (a circle) and the damped situation (clockwise spiral going toward the origin) in which the mass oscillates back and forth with decaying oscillations. Students’ predictions did not include the overdamped case, which would be graphically depicted by graphs in the phase plane that do not
spiral in toward the origin, and have at least one solution that falls along a straight-line. The fact that students did not initially predict the overdamped case sets up a useful pedagogical opportunity because the subsequent appearance of a SLS in the next part of the instructional sequence resulted in some surprise and interest on students’ part for further analysis (Rasmussen & Keynes, 2003). Indeed, we argue that, if a teacher wants students to reinvent important mathematical ideas, it is the responsibility of that teacher (with appropriate instructional materials) to foster in students the kind of curiosity and mathematical goals that have the potential to lead to the intended reinvention. Creating learning environments in which students encounter some surprise or perturbation is one way to do this.

Following student predictions, the teacher, with input from the class, used Newton’s law of motion to create the second-order linear homogeneous differential equation \( \frac{d^2x}{dt^2} + \left( \frac{b}{m} \right) \frac{dx}{dt} + \frac{k}{mx} = 0 \), where \( x \) is the distance the mass is from the rest position, \( k \) is the spring constant, \( b \) is the friction coefficient, and \( m \) is the weight of the mass. Fixing \( m = 1 \) and \( k = 2 \), this second-order equation was then converted into the following system of two first-order linear differential equations:

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= -2x - by
\end{align*}
\]

where \( \frac{dx}{dt} \) represents velocity and \( \frac{dy}{dt} \) represents acceleration. Next, the friction coefficient was allowed to vary from 0 to 3 using a java applet that continuously and dynamically displayed the corresponding changing vector field. Fig. 3 shows two snapshots of the changing vector field, one with spiraling solutions (a) and one without spiraling solutions (b). The friction coefficient, \( b \), in these two snapshots is 1 and 3, respectively.

Many students noticed that when the friction coefficient was approximately 3 there appeared to be a solution graph in the phase plane (see second quadrant of Fig. 3b) that heads directly toward the origin along a straight-line. It is at this point that students began to create and interpret analytic expressions for the observed and unexpected SLS.

Fig. 2. Spring-mass scenario and phase plane.

Fig. 3. Spiraling and non-spiraling solutions.
4.2. The model-of phase—creating and interpreting analytic expressions for SLSs

We characterize students’ mathematical work in the model-of phase in terms of the first two layers of activity in the emergent model heuristic—situational and referential activity. Situational activity involves students working toward mathematical goals in an experientially real setting. The slope of vectors (which serve as indicators of rate of change and the corresponding solutions to the rate of change equations) displayed in the phase plane for the spring-mass system of differential equations, together with students’ underlying imagery for the motion of the spring-mass scenario, serve as the experientially real starting point. In referential activity, students’ organizing activity with the differential equations function as a model-of their prior mathematical activity with the vectors and the observed slope of SLSs. Analytic expressions figure prominently in both situational and referential activity.

4.2.1. Situational activity

Because SLSs were not initially predicted by students, their observed presence in the phase plane led to a need to develop algebraic methods for determining the exact slope of the observed SLS, and students developed several ways to do this, not all of which were mathematically correct. For example, after fixing the friction coefficient \( b = 3 \) (see Fig. 3b), Tim and Larry conjectured (incorrectly) that \( \frac{dx}{dt} = \frac{dy}{dt} \) for the vectors falling along a straight-line in the phase. Their stated reason for doing this was a perceived proportional relationship between rate of change in position and rate of change of velocity.

\[
\text{Tim:} \quad \text{So I say that we say that its um, if I put the two equations equal to each other meaning the rate of change in position is equal uh the rate of change of velocity so}
\]

\[
\text{Larry:} \quad \text{Which would mean that its decreasing}
\]

\[
\text{Tim:} \quad \text{Proportionally}
\]

\[
\text{Larry:} \quad \text{Mmhmm}
\]

\[
\text{Tim:} \quad \text{Proportional to each other}
\]

In this excerpt Tim introduced the idea of a direct proportion between the two rates of change. We conjecture they did this because they perceived that in order to lose your oscillating solutions, you must have a “balance” between the two rate of change equations. After working through the algebra, they determined that \( y = -1/2x \) when \( \frac{dx}{dt} = \frac{dy}{dt} \). Based on this they concluded that the SLS has a slope of \(-1/2\). When presented in whole class discussion, however, this finding was rejected because it was not consistent with the vector field. A close inspection of Fig. 3b indicates the slope is approximately equal to \(-1\). Thus, in this instance, the observed vector field was a useful tool for determining the validity of Tim and Larry’s approach. As we argue next, however, using the observed vector field has its limitations.

In particular, several students noticed from the vector field that the straight-line of vectors appeared to fall on the line \( y = -x \). They then proceeded to set up the following proportion:

\[
\frac{dy}{dt} = \frac{-2x - 3y}{y} = -1.
\]

Solving for \( y \) in terms of \( x \) then yields \( y = -x \). Students then concluded that the slope of SLS is \(-1\). After some class discussion, however, students understood the limitation of this approach, as evidenced in the Hopi’s discussion board posting.

\[
\text{Hopi:} \quad \text{I did not know if I could assume that } y = -x. \text{ I also was not sure if the point was on the straight-line solution because we had not yet proved this before Wednesday. Now I know that there can be more than one straight-line solution and I would never been able to see this if I had not proved this algebraically or had gone by the very popular assumption that } y = -x
\]

Hopi, like many others, initially assumed that \( y = -x \) was the line on which a SLS lay. In her discussion board posting she alludes to an algebraic method that she now has to prove that the slope was \(-1\) with no a priori assumptions that this is the case. Hopi also commented at the initial surprise at finding that there were actually two SLSs, which was not readily apparent from the vector field. For her, the algebraic proof of two SLSs was satisfying because it finally allowed her to definitively determine the slope of any and all SLSs, without having to rely on “guess and test” methods.
While explicit discussion of the algebraic method itself is in the background of Hopi’s posting, Bryce’s discussion board posting brings to the foreground his way of thinking about his algebraic method that avoids guess and test.

**Bryce:** For a straight-line solution through the origin, y must be proportional to x. That is to say if \( \frac{\Delta y}{\Delta x} = (y_1 - y_0)/(x_1 - x_0) = \frac{y_1}{x_1} = m \). Since the slope is a constant any point \((x, y)\) on the line must have the same ratio \( m \). So \( y = mx \), which really should be no big surprise, but I guess the slope is not usually thought of as a proportionality constant.

Bryce’s comment that \( m \) is not usually thought of as a proportionality constant gives us pause to reflect on what, before this experience, was slope for Bryce? Was it simply some number, a static descriptor of a line? Unfortunately we do not have data on Bryce’s previous thinking about slope. However, it does appear that his work on this problem has deepened his thinking about slope as an invariant ratio characterizing any point on the line.

### 4.2.2. Referential activity

As stated earlier, in referential activity students’ organizing activity with the differential equations function as a model of their prior mathematical activity with the vectors and the observed slope of SLSs. That is, students’ prior algebraic methods for determining slope are enhanced and elaborated. In particular, these elaborations take on two distinct forms. First, a few students developed what we refer to as a dynamic interpretation of their algebraic method for finding the slope of SLSs. Second, essentially all students developed a method for determining the \( x(t) \) and \( y(t) \) equations for solutions that fall along a straight-line in the phase plane.

A dynamic interpretation of the algebraic method for determining slope of SLSs is characterized by imagining the motion or movement of vectors and the corresponding solution graph in the phase plane. We illustrate this dynamic interpretation with an excerpt from an end of the semester interview with Mario, who elaborates on how he thinks about the relationship between \( y = mx \) and \( dy/dx \).

**Mario:** Okay, for a straight-line solution to occur, um, \( y/x \) has to equal \( dy/dx \). Because, if the, okay, if the rate of change was different than that of the original slope that means it curves off because it is not going in the same direction. Its just saying, oh, if it was less than, it would pull it to one side and if it is greater than it would be pulling off to the other side. It has to be equal in order for a constant line to exist. So, that is the relationship I see between the two. Um, what was the question?

**Intrw:** What is the relationship between \( y = mx \) and \( dy/dx \)?

**Mario:** What is the relationship? Um, Okay, well, \( y = mx \), if you just change that around you get \( y = x \), \( y/x = m \), which is the slope and since \( dy/dx \) is the derivative of the top and one is the derivative of the bottom, it’s the rate of change, for one of each \([dy/dx = (dy)/(dx)/(dx/dt)]\). So, in a sense, um, the rate of change is like the future of \( y/x \), so in a sense that because, since its future, um, we can predict what the next \( m \) could be I guess. That is how I see a similar relationship between each other or one predicts each other. Like the \( y/x = m \) could predict \( dy/dx \), but then \( dy/dx \) would predict the next \( y/x \), or \( dy/dx \). So, they intertwine with each other

Mario first explains that the ratio of \( dy/dt \) to \( dx/dt \) has to equal that of \( y \) to \( x \) in order for a constant slope to exist. For Mario the relationship between these quantities is fluid and dynamic, as evidenced in his explanation for why the ratio of \( dy/dt \) to \( dx/dt \) must be equal to \( y/x \). He argues, by contradiction, that if this were not the case, then the movement or motion of the solution in the phase plane would “curve off” and “pull to one side.”

As Mario elaborated further, we gain insight into what causes the “pull,” that is, what causes the motion or movement. In particular, Mario says that it is the ratio of the rates of change that “predict the future” of the point \((x, y)\) in the phase plane. Mario’s idea appears to extend a ratio invariant conception of slope and considers the rate of change as a dynamic mechanism that in essence creates the solution graph. To clarify Mario’s dynamic interpretation, the SLS graph, starting at the initial condition \((x, y)\), gets “created” by the ratio of the rates of change.

We conjecture that Mario’s dynamic interpretation involves an iteration of the initial vector at the initial condition. This may well have origins in an earlier reinvention of Euler’s method for single differential equations (Rasmussen & King, 2000). Indeed, Mario argues that \( y/x \) predicts the “next” \( dy/dx \). This iteration, however, is more complex than the type of iteration children engage in when iterating number. For example, a child might create 10 by iterating 2 five times. In this case the quantity iterated does not change. However, in Mario’s iteration scheme, the vector that gets iterated changes size. If the SLS heads toward the origin successive vectors shrink in size. Conversely, if a SLS heads away from the origin, successive vectors expand in size. Further illumination of the students’ iterating schemes in which each successive vector is viewed in relation to the previous vector warrants further research.
The second form of students’ organizing activity with the differential equations that functions as a model-of their prior mathematical activity with the vectors and the observed slope of SLSs is in the creation of a method for determining the $x(t)$ and $y(t)$ equations for solutions that fall along a straight-line in the phase plane. Developing the $x(t)$ and $y(t)$ equations was relatively straightforward for students. For example, for the spring-mass system of differential equations (with $m = 1$, $k = 2$, and $b = 3$) students found that there are two SLSs, one with slope $-1$ and the other with slope $-2$, and hence the equations of the line along with these solutions lay are $y = -x$ and $y = -2x$, respectively. Thus, a straightforward substitution into the rate of change equations yields
\[
\frac{dx}{dt} = -x \\
\frac{dy}{dt} = -y
\]
for solutions along $y = -x$ and
\[
\frac{dx}{dt} = -2x \\
\frac{dy}{dt} = -2y
\]
for solutions along $y = -2x$. Solving these equations using separation of variables yields
\[
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix}
= \begin{pmatrix}
k_1e^{-t} \\
k_2e^{-2t}
\end{pmatrix},
\]
where $k_2 = -k_1$ and
\[
\begin{pmatrix}
x(t) \\
y(t)
\end{pmatrix}
= \begin{pmatrix}
k_3e^{-2t} \\
k_4e^{-t}
\end{pmatrix},
\]
where $k_4 = -2k_3$. In this case the value of the slope is the same as the value of the exponent for the $x(t)$ and $y(t)$ equations. This is not always the case, and students worked with other systems of differential equations for which the slope and value of the exponent were different. Students also readily used analytic expressions for the SLSs to justify why the graph of the SLSs in the phase plane would head toward the origin but never actually reach the origin.

Finally, once the $x(t)$ and $y(t)$ equations for two distinct SLSs are determined, the $x(t)$ and $y(t)$ equations for any solution not laying along one of these straight-lines can be determined by taking a linear combination of the two SLSs, as described in the Section 1. Students algebraically proved that this is indeed the case. However, a more challenging question, and one that is taken up in the next section, relates to what the graph of a solution in the phase plane looks like if the initial condition is not on one of the two SLSs.

4.3. The model-for phase—using analytic expressions for SLSs

In the previous sections we detailed how students’ organizing activity with analytic expressions functioned as a model-of their relevant mathematical reasoning about slopes of SLSs in the phase plane. Their mathematical work involved both the creation and interpretation of SLSs, including developing algebraic methods for determining the slope of all SLSs and the corresponding $x(t)$ and $y(t)$ equations. In the model-for phase, there is a shift from creating and interpreting to using SLSs as tools for accomplishing new goals. In particular, students’ organizing activity with the $x(t)$ and $y(t)$ equations for distinct SLSs begin to function as a model-for reasoning about the shape of all graphs in the phase plane. The collection of solution graphs in the phase plane (referred to as the phase portrait) represents an emerging new mathematical reality (for students). In conventional curricular approaches generating the graphs of solutions in the phase plane is typically relegated to technology. In contrast, we have found that students can use their self-generated analytic solutions for SLSs to determine, with reasons, the structured shape of all solutions in the phase plane.

4.3.1. General activity

In terms of RME, General activity facilitates a focus on interpretations and solutions independent of imagery integral to the original experientially real situation. Characteristic of such activity is a shift from creating to that of using prior results to achieve new goals (Zandieh & Rasmussen, 2007). As was the case with situational and referential activity, students’ subsequent mathematical activity involves qualitatively different ways of reasoning proportionally.
The examples of general activity that we tender involve students’ analyses of solutions to the system of differential equations $\frac{dx}{dt} = 2x + 2y$ and $\frac{dy}{dt} = x + 3y$. Students working with this system had correctly determined the two distinct SLSs, graphed the SLSs in the phase plane, and formed the general solution, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Students then used this result to determine the shape of the graph with initial condition $(1, 0)$, which does not lie on either SLS. This work led to the generalization for the shape of the graphs of all other solutions. That is, students’ work with the general solution now functioned as a model-for reasoning about new ideas, creating a new mathematical reality of the space of solutions in the phase plane.

In the following excerpt Anna begins by recounting for the whole class her initial (incorrect) reasoning about the shape of the graph. As illustrated in Fig. 4, Anna initially thought that the graph of the solution with initial condition $(1, 0)$ would be “pulled in” towards the straight-line solution with component $e^{4t}$ that lies along the line $y = x$.

Anna: My initial thought was that this [graph caving in toward $y = x$] is probably correct. I was thinking in a way well, since this function $e^{4t}$ grows faster with $t$ increasing then um, I was saying that this function would pull our solution to itself

Anna’s justification for the shape does not appear to coordinate the contributions from the two components of the general solutions, but rather focuses on the one component that has the greater exponent. At this point we claim that Anna is not engaging in proportional reasoning. However, as she continues to explain why she ultimately rejected this initial graph, she explains how she conceptualized the situation as the resultant of the two contributing SLSs.

Anna: Then the question, or, then was said, well, we still have this function $e^t$ grows faster with $t$ increasing then um, this grows faster [the component $e^{4t}$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, we still have this [the component $e^t$ $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$], so it kind of like pulls this function, so it kind of like pulls this function towards itself as well. Um, although the slope of this graph looks more like this [the SLS $y = x$]

As Anna revised her thinking, she sketched the graph of the solution as shown in Fig. 5. This new graph is distinguished from the previous graph by the fact that it does not curve in toward the SLS with the stronger “pull.” Instead, Anna’s new graph reflects a conscious coordination of the contributions from both components of the analytic solution.

Shortly thereafter Maria goes to the blackboard to further annotate Anna’s diagram and to explain her way of thinking about the resultant combination in terms of a multiplicative comparison of the two contributions of the SLSs.
Maria: So this $4t$ and this is $t$ [marks $y = x$ line as $4t$ and $y = -1/2x$ as $t$]. So its like $4t - t$, which is 3. This line [$y = -1/2x$] is pulling with a force of $t$ and this [$y = x$] is pulling with a force of $4t$. So, no matter how close it [graph of solution] gets to [$y = x$], there is always going to be at least one $t$ that is being pulled away [from $4t$]. This line is kind of pushing $4t$ that way. So, you see what I’m saying, you have this here like a string [traces over graph of solution with IC (1, 0)]. And you have somebody pulling with a force of one $t$ this way and somebody pulling with a force of four $t$. And no matter how, at what time, or whatever, there is always going to be a difference and uh, overall, or total, of $3/4$. There is going to be a fraction here that this line cannot get to [that other line, $y = x$]. I do not know how to say it

There is an important difference to note between Anna’s original argument for the shape of the graph and the two subsequent arguments. When Anna initially makes the claim that the graph will be “pulled” into the straight-line solution, she notes that the magnitude of the exponent corresponding to the straight-line solution $y = x$ is much larger, so she only attends to the “pull” of that component. The “pull” of the other component (the component corresponding to the straight-line solution $y = -1/2x$) fades into the background. In this sense, her reasoning might be characterized as univariate. That is, her reasoning focused on one quantity to the exclusion of the other quantity. In the second argument, Anna attends to the pull of both quantities, recognizing that however miniscule the contribution of the second component, it will still exert some force on the solution graph. This latter type of reasoning may be characterized as bivariate. The extent to which her bivariate reasoning is multiplicative in nature (as opposed to additive), however, is unclear.

Bivariate reasoning is further explicated by Maria, however in Maria’s argument she articulates a multiplicative relationship between the two straight-line solutions. “And you have somebody pulling with a force of one $t$ this way and somebody pulling with a force of four $t$.” We interpret Maria’s use of $4t$ and $t$ as a metonymy for $e^{4t}$ and $e^t$, respectively. She continues with, “And no matter how, at what time, or whatever, there’s always going to be a difference and uh, overall, or total, of $3/4$.” Thus, in a qualitative sense, these students are multiplicatively comparing the “pull” of the two distinct SLSs. Such multiplicative reasoning is a hallmark of proportional reasoning. In the final form of proportional reasoning we see how this multiplicative comparison is formalized in terms of a limit ratio.

### 4.3.2 Formal activity

We conclude the journey of students’ mathematical thinking by characterizing students’ Formal activity. As described by Gravemeijer (1999), Formal activity involves reasoning with conventional symbolism, which is no longer dependent on the support of models-for mathematical activity. In our particular case, Formal activity involves using the SLS components without having to revisit or unpack the meaning of the components. The following example continues the excerpt of Anna’s recounting of her reasoning.

To further support her conclusion that the graph of the solution with initial condition $(1, 0)$ does not get pulled in toward the line $y = x$ (recall that this was Anna’s initial idea and thus it makes sense that she would want to garner as much evidence to the contrary as possible), Anna computes the $\lim_{t \to \infty} \frac{y(t)}{x(t)}$. She also computes this same limit for time $t$ going to minus infinity. We view this computation as a formalization of the more qualitative multiplicative argument put forth by Maria.
Anna: We also solved to make certain what it is [the graph], that it looks something like this [referring to her revised graph shown in Fig. 5], we also solved limit of um. I forget the word, $y(t)$ over $x(t)$

Teacher: Ratio?

Anna: Yeah, ratio. And after calculations, it [the limit of $y(t)$ over $x(t)$ as time approaches infinity] was equal to 1. And then, um we did the same [compute the same limit], but for $t$ approaching negative infinity. And that was $-1/2$. From that, I realized that um no matter where we start, where our initial condition is, if $t$ approaches a positive infinity, the graph tries to look, I will rephrase it, the graph does not try to look, the um, the slope of the graph, because this $[y(t)/x(t)]$ represents the slope, will look more like, or get closer to 1 and if $t$ approaches negative infinity, then the slope of the graph will look closer to $-1/2$

Noteworthy is Anna’s careful attention to language, which we take to be indicative of formalizing one’s informal or qualitative reasoning. “If $t$ approaches a positive infinity, the graph tries to look, I will rephrase it, the graph does not try to look, the um, the slope of the graph, because this $[y(t)/x(t)]$ represents the slope, will look more like, or get closer to 1.” Anna intentionally avoids saying “the graph tries to look,” which perhaps carries for her the connotation that the graph is “pulled” toward $y=x$, exchanging it for language that reflects that the conclusion of the limit pertains to the slope of the graph.

In addition, using the limit of the ratio of $y(t)/x(t)$ seems particularly powerful for Anna. In particular, she remarks on the generality of her conclusion based on her limit argument. “From that, I realized that um no matter where we start, where our initial condition is…” Anna observed that her limit argument gives the ultimate slope for any initial condition as time approaches positive and negative infinity. That is, Anna’s reasoning with the ratio using limits now functions to structure the space of all solution graphs in the phase plane, giving rise to the new mathematical reality of the structured phase portrait.

5. Concluding remarks

This article makes a contribution to the RME emergent model heuristic. In particular we demonstrated how analytic expressions can underpin the entire model-of/model-for transition. As we noted earlier, the case of differential equations offers a prototypical example for RME-inspired work at the undergraduate level to support learners in the creation of new mathematical realities. The analysis also points to students’ increasingly sophisticated ways of reasoning with proportions. Such insights into student thinking can be useful for teachers in their difficult task of making sense of what students say and do, and for being proactive in supporting their intellectual growth.

On a more pragmatic level, we are interested in how retrospective analyses such as this one might inform the revision of the instructional materials. For example, we are currently considering the advantages and disadvantages of choosing parameter values that result in students creating algorithms in which they implicitly assumed the slope of the SLS was $-1$. On the other hand, raising awareness of the limitation of this approach seemed useful for many students. On the other hand, because of time constraints it would have been more efficient if the specific parameter values used in the student materials were such that students could not easily observe a slope of $-1$. In addition, the java applet we designed that displays vectors fields also displays values for $x$, $y$, $dx/dt$, and $dy/dt$. The presence of these values may have contributed to students observing a slope of $-1$ and then using this value as a given rather than deriving this value.

We are also encouraged by the kind of dynamical interpretation that Mario generated. This was a particularly rich and powerful way of coordinating quantities. We look forward to revising the instructional materials to further engage students in imagining the relationship between successive vectors along a SLS, and to conducting future research that examines how this type of dynamic reasoning relates to the broader literature on proportional reasoning. For example, we are curious how a dynamic interpretation of the relationship between $y = mx$ and $dy/dx$ might relate to the concept of unitizing. According to Lamon (1994), unitizing is “the ability to construct a reference unit or a unit whole, and then to reinterpret a situation in terms of that unit. . .[which] involves the progressive composition of units to form increasingly complex structures” (p. 92). In the case of children’s activity of counting, the reference unit that is iterated to create a 10, for example, is invariant. However, in the case of vectors along a straight-line solution, the initial vector that one chooses to iterate is not invariant, its length changes. As Mario noted, “$y/x = m$ could predict $dy/dx$, but then $dy/dx$ would predict the next $y/x$.” The successive lengths of vectors depicting $dy/dx$ change as the values of $x$ and $y$ change. Similar to how a 10 can be created by iterating a two, a SLS can be created by iterating a vector, the difference being that the two remains invariant while the length of the vector does not.
Appendix A

Most, if not all, differential equations textbooks solve systems of linear differential equations using techniques from linear algebra. In this method, the eigenvalues are computed first and then the corresponding eigenvectors are determined. For example, consider the system of linear differential equations given by

\[
\begin{align*}
\frac{dx}{dt} &= 3x + 5y \\
\frac{dy}{dt} &= 2x + 6y
\end{align*}
\]

This system can be rewritten into matrix form as \( X' = AX \) where \( X' = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} \), \( A = \begin{bmatrix} 3 & 5 \\ 2 & 6 \end{bmatrix} \), and \( X = \begin{bmatrix} x \\ y \end{bmatrix} \). To find the eigenvalues of the matrix \( A \), compute the determinant to obtain the following:

\[
\begin{vmatrix}
3 - \lambda & 5 \\
2 & 6 - \lambda
\end{vmatrix} = (3 - \lambda)(6 - \lambda) - (5)(2) = \lambda^2 - 9\lambda + 8.
\]

Setting this determinant equal to zero produces the characteristic equation of the system. The solutions to \( \lambda^2 - 9\lambda + 8 = 0 \) are the eigenvalues \( \lambda_1 = 8 \), \( \lambda_2 = 1 \).

The next step is to find the eigenvectors associated with each eigenvalue. Plugging \( \lambda_1 = 8 \) into the characteristic equation \( A - 8I = 0 \) produces the matrix equation

\[
\begin{bmatrix}
-5 & 5 \\
2 & -2
\end{bmatrix}
\begin{bmatrix}
s \\ t
\end{bmatrix} =
\begin{bmatrix}
0 \\ 0
\end{bmatrix}.
\]

The eigenvector \( v_1 = \begin{bmatrix} s \\ t \end{bmatrix} \) must satisfy the equations \(-5s + 5t = 0\) and \(2s - 2t = 0\). Both equations produce the equation \( s = t \), so any multiple of the vector \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \) will work. Following a similar procedure for the eigenvalue \( \lambda_2 = 1 \), the corresponding eigenvector is \( v_2 = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \). The general solution to this system of linear differential equations is a linear combination of the two eigensolutions. In this example, the general solution is

\[
X(t) = C_1e^{8t}v_1 + C_2e^{2t}v_2 = C_1e^{8t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2e^{2t}\begin{bmatrix} 5 \\ -2 \end{bmatrix}.
\]

References


