Notation Guide for Precalculus and Calculus Students

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Contents

NOTE: The items listed in blue type and red type are the ones that are worth points on your tests. The items in blue give advice that, if not followed, will cost you points. The items marked with the word "**Incorrect:**" show examples of incorrect notation for which points will also be lost. Replace this poor notation with the corresponding notation marked as "**Correct:**".

This doesn't mean, however, that you should ignore the other advice! Mathematical expressions that are technically correct can still be ugly, so while you may not lose points for them, you're not doing yourselves any favors by insisting on using them. Even when two or more notations for the same thing are correct, they may be labeled by "**Preferred:**" or "**Not Preferred:**". While these are both technically correct, you should try to avoid notation that is "not preferred". Having said that, you won't lose points for using "not preferred" notation.

1 Introduction

Most of us at one point or another in our education have to study a foreign language. We have to learn new vocabulary words and learn to spell them. We must conjugate verbs. We often struggle to adapt our thinking to different sentence structures and idiomatic expressions.

Math, too, is like a foreign language. Expressions are words and equations are sentences. There are precise rules for notating mathematical thought. Unfortunately, students may have received the mistaken impression early on in the educational process that writing down mathematics is just a means to an end. After all, unlike a foreign language for which communication is the obvious goal, elementary mathematics seems to have as its purpose the acquisition of the "answer". As long as the number you get matches the answer in the back of the book, your job is done.

Without belaboring the point too much, the work involved in getting to the answer is at least as important as the answer itself. The focus of calculus and higher-level mathematics is the method. Having said that, then, it is of critical importance that students begin to learn good notational habits (or, stated another way, to reverse bad notational habits) to communicate such work with maximal precision. It is unfortunate that many teachers at lower grade levels miss the opportunity to instill such habits.

Of course, placing importance on notation presents a host of problems to professors and students.

Professors try to use correct notation in their lectures and they want their students to do the same. They become frustrated when students fail to emulate their notational style. Their work in grading is doubled if they make any effort at all to try to make students accountable for notation.

Students often come to calculus with less than adequate preparation from their previous classes and so they experience frustration when they lose points on exams for their work despite getting the correct answer. Even more aggravating is when the professor assumes that their work is "sloppy" or "lazy", whereas the students believe that they are working quickly and efficiently.

What constitutes "good" or "bad" mathematical notation? It is true that the conventions for mathematical notation do change over time. A math book written fifty years ago is likely to look somewhat different than a book written today. Also, there are variations in notation due to personal preference: different authors often prefer one way of writing things over another due to factors like clarity, concision, pedagogy, and overall aesthetic. Nevertheless, there are certain practices which have become fairly standard and there are other practices which are universally considered incorrect as well. This guide serves to educate the precalculus or calculus student about the generally accepted standards of correct and incorrect mathematical notation.

The most general advice is to watch what your professor writes. Take good notes and then use them when working homework or practice problems to make sure that the way that you write agrees with the way your professor writes.

2 Expression versus equation

Before we dive into the math, there is some vocabulary that needs to be settled once and for all. It will be important in what follows to recognize the difference between an "expression" and an "equation" since we will use these terms in almost everything we do from here on out.

Problem:

Students often confuse these two terms and as a result, they confuse the methods used for dealing with them.

Solution:

Understand the definitions of expression and equation. An expression is a mathematical quantity. An equation is the relationship of equality between two expressions. (Note that an inequality is also a relationship between two expressions, but inequality is a term that is rarely misused.)

Think of an expression as a noun. In a sentence, it must be used as such.

Incorrect:

It follows that $f(x)$.

Correct:

It follows that $f(x)$ is an increasing function.

On the other hand, an equation is a full sentence. The equal sign functions as a verb.

Incorrect:

It follows that $f(x)$ is $= 2x$.

The symbol $=$ means "is equal to", so the last sentence actually says, "It follows that $f(x)$ is is equal to $2x$." Instead, just write

Correct:

It follows that $f(x) = 2x$.

3 Handwritten math versus typed math

While your book is typeset very neatly, you will be required to write all your math by hand. Writing math presents a few common problems. This section is less about correct versus incorrect and more about just being neat.

3.1 Numerals

Problem:

Numbers can be confused for other things. The number zero, 0, looks just like the letter O. The number one, 1, is often written as a simple vertical line, $|$, which is also a symbol used for other things (like absolute value). Written quickly, a 2 can look like a Z. A 5 looks like S. I've seen 7's that look like right parentheses,), or right angle brackets, $>$. In a case of a number being confused for another number, a 9 can look like a 4.

Solution:

Generally, the solution is just to write clearly. Ninety-nine percent of all such ambiguities can be resolved with good handwriting.

Zero: This one usually doesn't cause too much trouble in calculus since it won't often be confused for the letter O in expressions and equations. Sometimes I'll write my zeros like $\emptyset$, but I don't do so consistently. (This is actually the symbol for the "empty" set, but in calculus I see no harm in using it for zero. Early character displays for computers used this "slashed zero" precisely because of this confusion with the letter O.)

One: If there is no potential for confusion, a single vertical slash, $|$, will communicate what you need. If there is possibility for confusion, write it like 1, with a flag at the top and a line across the bottom. Again, I don't follow any particular rule here consistently.

Seven: With good handwriting, this should never be a problem. A long time ago, I got into the habit of writing my sevens with a little slash in the middle, like so: 7. But this isn't strictly necessary.

3.2 Letters

Problem:

Just like numbers, letters can be misinterpreted if written incorrectly.

Solution:

Once again, the solution is usually just to write clearly. But there are a few cases where extra caution is required.

Lower-case l: This shouldn't come up much, but when it does, you should use a cursive ℓ so it's not confused with the number 1.

Lower-case t: Notice that in most fonts, the letter t has a little curve at the bottom. On the other hand, many people write t's that look like this:**†**. The difference between **†** and the plus sign, $+$, is of course that in **†** the vertical line is probably a bit longer—that is, unless you are writing quickly, in which case it can be impossible to tell the difference. Just get into the habit of writing t's the same way they are typed: with a little curve at the bottom. It is my experience that this practice is almost universal among mathematicians and scientists.

Lower-case y: Since x and y are so common, they appear together frequently. Just make sure when drawing the y that the smaller bar doesn't accidentally cross the longer bar; otherwise, it can look like an x. It's easy to make this mistake when you're in a hurry.

4 Use of calculators

Problem:

Students are far too dependent on calculators to do the job.

Solution:

Learn when calculators are helpful and when they are not. Calculators are good for checking your work and giving you a little extra boost of confidence when you are on the right track. They are not good as a substitute for understanding how to do algebra.

You might be asking, "Why is a section on calculators in a guide on notation?" Because this is primarily a notation guide, I won't go into the issue of calculators as deeply as I might otherwise. Suffice it to say that using a calculator instead of showing work will cause notational and mathematical errors. The most pressing is illustrated by the following example.

Suppose that you know the initial population of a city is 100,000 people and in one year, the population has increased to 110,000. Given that the population is growing exponentially, find the population in five years.

The first thing to do is see that the function describing exponential growth is given by

$$P(t) = P_0 e^{kt}.$$

where P_0 is the initial population, k is a growth constant, and t is time measured in years. In this example, we are told that $P_0 = 100{,}000$ and also that when $t = 1$, $P(1) = 110{,}000$. Hence, the formula says

$$110{,}000 = 100{,}000 e^{k(1)} = 100{,}000 e^{k}.$$

The constant k is unknown, but we can solve for it.

$$\begin{aligned} \frac{110{,}000}{100{,}000} &= e^{k} \\ 1.1 &= e^{k} \\ \ln 1.1 &= k \end{aligned}$$

Now here's where the error might occur. If we use a calculator to evaluate this, we get

Incorrect:

$$k = 0.0953101798043249.$$

Why is this incorrect? In fact, it seems correct to a high degree of accuracy. The problem is that $\ln 1.1$ has a non-terminating, non-repeating decimal part, so no number of digits we write down will be the exact right answer for k. In and of itself, this error isn't so egregious, but usually students take it one step further. For ease of working with this value, one might try to round it.

Incorrect:

$$k = 0.095$$

Now let's see what happens when we try to solve the problem with the rounded number. Knowing k, we now write the function as

Incorrect:

$$P(t) = 100{,}000e^{0.095t}.$$

All we need to do to find the population after five years is plug in $t = 5$.

Incorrect:

$$P(5) = 100{,}000e^{0.095(5)},$$

which we plug into our calculator to obtain

Incorrect:

$$P(5) = 160{,}800.$$

This time, let's leave k as $\ln 1.1$ and do it the right way.

Correct:

$$\begin{aligned} P(t) &= 100{,}000e^{(\ln 1.1)t} \\ P(5) &= 100{,}000e^{(\ln 1.1)(5)} \\ &= 100{,}000e^{5 \ln 1.1} \\ &= 100{,}000e^{\ln (1.1)^5} \\ &= 100{,}000\,(1.1)^5 \end{aligned}$$

This is an *exact* answer, and is the answer you should give as your final answer. Having said that, this is a problem about populations, and the number $100{,}000\,(1.1)^5$ isn't all that helpful to the census bureau. At this point you can, if you wish, put this into a calculator and obtain

Correct:

$$P(5) \approx 161{,}051.$$

Notice the margin of error here. Using our rounded value of k, we were off by 251 people! Now as a percentage, that's really only about 0.16% of the total, so it doesn't seem like that much when we look at it like that. The point here, though, is that there is no need to introduce errors when simple algebra will give us an exact answer.

Also note the use of the symbol $\approx$ for "approximately equal to" above. This is appropriate, whereas

Incorrect:

$$P(5) = 161{,}051$$

is not. This is not the exact answer (although it is rounded off to the nearest person, which is really the only thing that matters since people come in whole units), so it is not correct to use the equal sign.

The lesson here is, do not use your calculator to substitute numbers for algebraic expressions. The only exception is your final answer, and even that is better left as an exact expression. You only need a numerical answer if you are solving a real-world problem in which the algebraic expression would be of no use.

Getting a numerical answer does allow you to check if the result you obtained is a reasonable answer. For example, in the above example, if we had obtained 45 for the population, we would know we made a mistake, and that can be valuable. So, once again, calculators can help you ensure that you have the right answer, but they should not be giving you the answer you find through your work on paper.

5 General organizational principles

5.1 Legibility of work

Problem:

When working problems on homework or tests, the focus is too often on simply getting the answer. Therefore, the work above the answer is likely to be scrambled and illegible. The area on the page in which you show your work is often littered with formulas, expressions, equations, and sketches, some of which are relevant to your answer and much of which isn't.

Solution:

As always, write neatly. It only takes a few extra minutes and it helps everyone. It obviously helps the grader since he or she will not have to stare at your paper as long to figure out what you've done. But it also helps you since when you go back to check your work, you will not have to stare at your paper as long to figure out what you've done.

Also, try not to view your work space as "scratch paper". Remember that the grader is evaluating your work as it appears on the page, so put your best foot forward. If you need to jot down notes or do some experimental calculations, be sure that it's clear on your page that these are not part of the main "flow" of your argument. Do it at the bottom of the page or off to the side. Draw a line separating this stuff from the work you want graded, and then label that section with "scratch work" or "do not grade". Alternatively, do such work on the back of the page or another piece of scratch paper.

If you end up trying something that doesn't work, make sure you cross it out somehow. Otherwise, the grader will judge it as if you meant to have that as part of your answer. (While I'm recommending that you cross out such work, maybe by drawing a big X though it, I'm not necessarily recommending that you erase it altogether. It might be helpful for you to refer to as you work the problem so you don't make the same mistakes multiple times, or so you don't accidentally waste time trying the same incorrect lines of attack.)

5.2 Flow of work

Problem:

This is also a legibility problem, but it stems from a lack of logical progression from one step to the next. There are many manifestations of this problem. Formulas might be scattered all over the page. Complicated expressions might be written close to the bottom of the page and then simplified in steps that are written all over the page in places not contiguous to the original expression.

Solution:

The thing to remember is that we read (at least in English) from left to right and from top to bottom. Your math should "read" the same way. If you are simplifying an expression, start at the top left with the expression, then write an equal sign, =, and then carry out one step. Then write another equal sign and carry out the next step. Continue doing this until you arrive at the answer. Then indicate that you have arrived, either by circling it or putting a box around it, or by transferring that answer to an appropriate space at the bottom of the page designated for writing down your answer.

If you are moving down the page, the preferred placement of the equal sign is at the beginning of the subsequent line as opposed to the end of the previous line.

Not Preferred:

$$\frac{x^2 - 2x + 1}{x^2 - 1} = \frac{(x-1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} =$$
$$\frac{x-1}{x+1}, \qquad x \neq 1$$

Preferred:

$$\begin{aligned} \frac{x^2 - 2x + 1}{x^2 - 1} &= \frac{(x-1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} \\ &= \frac{x-1}{x+1}, \qquad x \neq 1 \end{aligned}$$

(See section 7.4, *Domain matching*, for more about the condition $x \neq 1$ appearing in the above calculations.)

If you are solving an equation, then each time you do something to both sides of the equation, show what you are doing. Then write the equation that results from the operation below the first. Since you're working with an equation, each line should only contain one equal sign. The following example shows what can go wrong.

Incorrect:

$$\begin{aligned} x^2 + 2 &= 11 \\ -2 \quad &\;\; -2 \\ = \quad \sqrt{x^2} &= \sqrt{9} \\ = \quad x &= \pm 3 \end{aligned}$$

The incorrect work suggests that the 11 at the end of the first line is equal to the x^2 at the beginning of the next equation.

Correct:

$$\begin{aligned} x^2 + 2 &= 11 \\ -2 \quad &\;\; -2 \\ \sqrt{x^2} &= \sqrt{9} \\ x &= \pm 3 \end{aligned}$$

Just remember that an equal sign doesn't mean, "This is the next step." It can only be used to express precise mathematical equality between two expressions.

Also important: do not write an equal sign when you only have stuff on one side of the "equation". If you get stuck in a dead-end calculation, don't just leave an equal sign floating at the end with nothing to the right of it. Erase the equal sign so that the expression right before it is the last expression of the calculation.

5.3 Using English

Problem:

We are so used to worrying only about the answer that we often neglect the reasoning that leads to the answer. Our workspace is littered with formulas, expressions, equations, and sketches. All of these things are good things, but where is the reasoning that justifies all of this? Where are the words that explain what we are doing and tie everything together?

Solution:

Use words! In fact, use full sentences! Sure, there's a certain amount of basic calculation we do: solving equations, simplifying expressions, performing simple arithmetic to get from one step to the next. We do not need to justify every little thing we do. But when we arrive at the "crucial step", the one that involves the main concept being practiced or tested, it must come with some kind of explanation. If nothing else, think of it as self-preservation: when you explain your reasoning, you maximize your potential for partial credit. If the teacher knows what you're trying to do, you may get an extra point for explaining it correctly even if you made an error in the computation.

Problem:

After learning new mathematical symbols, students become quickly enamored of them and start using them as substitutes for actual words.

Solution:

You should never use symbols in exposition when full English sentences are required. By exposition, I mean the sentences you write to explain your reasoning or to state facts, not the computational steps you use to derive your answer. (Obviously those will use lots of mathematical symbols.)

Incorrect:

We see that $\exists$ an $x \ni x > 0 \therefore y$ must be +

Correct:

We see that there exists an x such that $x > 0$. Therefore, y must be positive.

First a note about the incorrect example. The symbol $\exists$ does mean "there exists" and $\therefore$ does mean "therefore". (The $\ni$ is sometimes used for "such that", but its use is far less frequent.) So why can't we use them? Clearly the example above is over the top, but it would still be wrong even to use just one of these symbols without the others. The reason is that these symbols are inappropriate for formal exposition. You'll never see a textbook use these symbols (at least you shouldn't see one). Why are they around, then? Their function is limited to the blackboard, where their use is lamentable but necessary to keep lectures moving at a reasonable pace. Even at that, you won't see me using these symbols in my precalculus and calculus classes. (If you study formal logic, you will see frequent and appropriate use of the "logical quantifiers" $\exists$ for "there exists" and $\forall$ for "for all"). But under *no* circumstances should $+$ *ever* be used to replace the word "positive", nor should $-$ be used to replace "negative". Even more awful are such abominations as "$+$ ive" and "$-$ ive".

In the correct example, note that it was not necessary to write out "such that x is greater than zero", although it would not have been incorrect to do so. Elementary symbols that express concepts from basic arithmetic are actually *more* clear than the excess verbosity required to express the same concept in words.

Problem:

As it is wrong to use symbols in exposition involving full English sentences, it is equally wrong to put words in the middle of expressions and equations.

Solution:

In mathematical expressions and equations, only numbers, letters, and symbols should be used. If you need to explain something, do it off to the side or underneath, but not in the math itself.

Incorrect:

$$\frac{(x-1)(x+3)}{x^2+4} = \frac{(\text{negative})(\text{positive})}{\text{positive}} = \text{negative}$$

Correct:

$$\frac{(x-1)(x+3)}{x^2+4}$$

Since this is the product of a negative number

and a positive number, divided by a positive number,

the result is negative.

6 Precalculus

It would be quite difficult, if not impossible, to catalogue all the possible errors of notation that one might see. (Even when I think I've seen them all, I keep seeing new ones.) But here are some of the more common problems that many of us have hung on to for far too long.

6.1 Multiplication and division

In elementary school we learned the symbols $\times$ for multiplication and $\div$ for division.

Problem:

The symbol $\times$ and the variable x are usually indistinguishable.

Solution:

Just don't use $\times$ for multiplication.

Incorrect:

$$x \times \sin x$$

Sometimes you can use the dot $\cdot$

Not Preferred:

$$x \cdot \sin x$$

But usually you don't even need that. Use *concatenation* instead.

Preferred:

$$x \sin x$$

If you're multiplying two numbers together, obviously you can't just use concatenation.

Incorrect:

$$5\ 6 = 30$$

(I don't honestly think anyone would actually make that mistake!) Here the dot is an option.

Correct:

$$5 \cdot 6 = 30$$

Or use parentheses.

Correct:

$$(5)(6) = 30$$

But you don't need parentheses and a dot.

Not Preferred: (and bordering on incorrect)

$$(5) \cdot (6) = 30$$

Also if you are trying to write, say, $x^2 + 4$ times $x - 3$, it is wrong to write

Incorrect:

$$x^2 + 4 \cdot x - 3 \text{ for "}(x^2 + 4) \text{ times } (x - 3)\text{".}$$

since $x^2 + 4 \cdot x - 3$ actually means $x^2 + 4x - 3$. In this case the dot did you no good since you are multiplying expressions that consist of more than one piece. Use parentheses.

Correct:

$$(x^2 + 4)(x - 3) \text{ means "}(x^2 + 4) \text{ times } (x - 3)\text{"}$$

Problem

The division sign ÷ is useless and leads to errors.

Solution:

Don't use the division sign ÷. Use a fraction instead.

6.2 Fractions

Problem:

Sometimes it is not clear which elements of an expression are in the numerator and which are in the denominator.

Solution:

Avoid using a forward slash to indicate fractions except in a few specific cases.

Incorrect:

$$1/5x + 3$$

Either

Correct:

$$\frac{1}{5x+3},$$

or

Correct:

$$\frac{1}{5x} + 3.$$

depending on which you mean. I suppose you could use parentheses to clarify:

Not Preferred:

$$1/(5x + 3)$$

or

Not Preferred:

$$(1/5x) + 3,$$

but why bother when a horizontal line is so much more clear?

For rational numbers—that is, fractions where the numerator and denominator are just integers—then it doesn't really make much difference.

Correct:

$$1/2 \qquad 5/4 \qquad 13/201$$

or

Correct:

$$\frac{1}{2} \qquad \frac{5}{4} \qquad \frac{13}{201}$$

There are cases where it's okay (although still not necessarily desirable) to use a forward slash. If there's only a single term in the denominator, then there isn't much ambiguity.

Not Preferred:

$$x^3/\sqrt{x} \qquad\qquad 3/x^2$$

Preferred:

$$\frac{x^3}{\sqrt{x}} \qquad\qquad \frac{3}{x^2}$$

There are a few cases in which a forward slash can be slightly more clear than a horizontal bar, although you are never forced to use a forward slash. The first is when you have fractions in the numerator and denominator of a fraction.

Correct:

$$\frac{1/x}{3/x^2}$$

Of course, one can use all horizontal bars to write this as

Not Preferred:

$$\frac{\frac{1}{x}}{\frac{3}{x^2}}$$

but maybe it's less "pretty" since the bars are all roughly the same length. Write this instead as

Preferred:

$$\frac{\;\frac{1}{x}\;}{\frac{3}{x^2}}.$$

When you need to be careful is when one of these expressions is a fraction and the other isn't.

Incorrect:

$$\frac{x}{\frac{3}{x^2}}$$

In this case, it is impossible to know if this is the fraction

Correct:

$$\frac{x}{3/x^2}$$

or

Correct:

$$\frac{x/3}{x^2}.$$

(Be sure you see that these really mean two different things.) You can still use horizontal bars here, but you have to make sure that one of the bars is longer than the other so there's no ambiguity.

Correct:

$$\frac{x}{\frac{3}{x^2}}$$

Or use parentheses to remove the ambiguity.

Correct:

$$\frac{x}{\left(\frac{3}{x^2}\right)}$$

Another case of forward slashes used correctly involves exponentiation. When a fraction is in the exponent, it is often seen written with a forward slash.

Correct:

$$e^{x/2}$$

Of course, it is fine to write this with a horizontal bar.

Correct:

$$e^{\frac{x}{2}}$$

The only reason one doesn't see more of the latter is that in printed mathematics the former can be written in one line without disrupting the spacing of the line above it. In your hand-written work, this isn't a problem.

One must also be cautious about writing a fraction next to an expression not in a fraction.

Incorrect:

$$\frac{2}{3 \quad x}$$

Is this $\frac{2}{3}$ times x? Or did the bar not quite make it far enough, meaning that it would be $\frac{2}{3x}$? Being more cautious we would write

Not Preferred:

$$\frac{2}{3}x$$

or better yet,

Preferred:

$$\frac{2x}{3}$$

Usually it is better to make sure that we place everything squarely in the numerator or denominator. The common exception to this rule is polynomials. Since we think of polynomials as powers of x with coefficients, is is better notation to put a fractional coefficient in front of the power of x.

Not Preferred:

$$\frac{x^2}{2} - \frac{3x}{4}$$

Preferred:

$$\frac{1}{2}x^2 - \frac{3}{4}x$$

6.3 Functions and variables

Problem:

Students often use the lower-case and the upper-case versions of variables interchangably.

Solution:

Leave letters the way they're written in the problem. If you are introducing a name for a variable or function, stick to it throughout the computation. It may not seem like it matters, but it does. For example, in calculus, if $f(x)$ is a function, then it is common to use $F(x)$ to denote its antiderivative. (Don't worry if you don't know what this means yet; it's just an example of how lower-case f and upper-case F can mean totally different things in the same problem.)

Problem:

Sometimes a letter is used to mean one thing in one part of the calculation and then the same letter is used for something else later in the same calculation.

Solution:

If you need to introduce another function or variable, be sure to use a different letter for it.

Problem:

A function is expressed in terms of an independent variable. One can substitute values for this values and evaluate the function with numbers or other expressions. But the notation changes when this happens.

Solution:

Do not connect with an equal sign a function expressed with a variable and the same function after plugging in a number.

Incorrect:

$$\begin{aligned} f(x) &= x^2 - 4x \\ &= 2^2 - 4(2) \\ &= -4 \end{aligned}$$

Correct:

$$\begin{aligned} f(x) &= x^2 - 4x \\ f(2) &= 2^2 - 4(2) \\ &= -4 \end{aligned}$$

6.4 Roots

Problem:

The bar over the radical doesn't always "cover" everything that's supposed to be inside the root.

Solution:

Extend the bar to make sure it is long enough. Also follow the advice in section 6.10, *Order of functions*, to avoid this problem.

Incorrect:

$$\sqrt{x\ y\ }z$$

Is the z supposed to be inside or not? Write

Correct:

$$\sqrt{xyz}$$

if that's what you mean, or if the z is not part of the root, then

Not Preferred:

$$\sqrt{xy}\ z,$$

but better,

Preferred:

$$z\sqrt{xy}.$$

6.5 Exponents

Another recommendation involves negative exponents.

Not Preferred:

$$(-1/3)\,x^{-4/3}$$

Preferred:

$$\frac{-1}{3x^{4/3}}$$

This is due to the fact that if we have to evaluate such a function by plugging in a specific value of x, it is much easier to work with positive exponents. After all, if there were a negative exponent left in the expression, we would move it to the other half of the fraction before evaluating anyway.

Fractional exponents are a bit on the ugly side, too, but usually there is not much can be done about them. In the event that the fraction is 1/2, one can use square root instead.

Not Preferred:

$$x^{1/2}$$

Preferred:

$$\sqrt{x}$$

I reiterate here that I am not suggesting the expression $x^{1/2}$ is never to be used. It's just when you're simplifying your final answer, my recommendation is that you change it to a square root.

But for all other fractions, you can decide which you think is prettier. (They're all fairly ugly, though.)

Correct:

$$x^{3/4} = \sqrt[4]{x^3} = \left(\sqrt[4]{x}\right)^3$$

Unless the power is 1/2, I personally prefer to leave all fractional exponents written as fractions.

There is one case that requires extra caution. If you do write a fractional exponent with a root symbol, the following could happen:

Incorrect:

$$x\sqrt[4]{x}$$

It's not clear if you meant to write

Correct:

$$(x)\left(\sqrt[4]{x}\right)$$

or

Correct:

$$\left(x^4\right)\left(\sqrt{x}\right).$$

See it all depends on where you place the 4. So be sure that if the root is a cubed root, fourth root, etc., that the number indicating this is buried deep inside the "v"-shaped part of the root symbol where it can't be mistaken for an exponent of anything to the left of it.

6.6 Inequalities

Problem:

When using inequalities to express intervals, one has to be careful about the order of terms.

Solution:

Let's suppose we are solving for x in the inequality $x^2 > 9$. Then x can be greater than 3 or less than -3. But you cannot write

Incorrect:

$$3 < x < -3,$$

nor can you write

Incorrect:

$$-3 > x > 3.$$

Even though each individual inequality is correct, $3 < x$ and $x < -3$, inequalities are also transitive. This means that if we write $a < b$ and $b < c$, then it should follow that $a < c$. So in this case, we are also saying that $3 < -3$, which is obviously not true.

The correct way to express this is

Correct:

$$x > 3 \text{ or } x < -3.$$

If we are solving the inequality $x^2 \leq 9$, then this isn't a problem. Now the x values that satisfy this inequality lie in a range between -3 and 3, so we can write

Preferred:

$$-3 \leq x \leq 3.$$

While

Not Preferred:

$$3 \geq x \geq -3$$

is technically correct, we prefer to see the numbers in increasing order.

Problem:

Sometimes infinity, ∞, appears in inequalities. There is a right way to do this and a wrong way to do this.

Solution:

The inequality

Incorrect:

$$x < \infty$$

should not be used. It's not really that it is untrue; after all, every number is finite. But since every number is finite, there's no reason for this ever to be written. (While this is true for variables, it is not true for certain limits and integrals in calculus. See sections 8.2 and 10.5 for more on this.)

Even worse is

Incorrect:

$$x \leq \infty$$

Again, still technically a true statement, but since no number is ever equal to infinity—the phrase "equal to infinity" is complete nonsense—it makes no sense even to suggest that the "less than or equal to" could ever be "equal to".

Technically speaking, by what was said above, the following *should* be incorrect, but since it is seen in books all the time, I have to concede that it is permissible to use infinity in an inequality when you want to express a range of possible values of your variable. For example,

Not Preferred:

$$2 < x < \infty$$

or

Not Preferred:

$$-\infty < x \leq 5$$

In the former, you are indicating any value of x bigger than 2 and in the latter, any value of x equal to 5 or less. But then why not just write

Preferred:

$$x > 2$$

and

Preferred:

$$x \leq 5$$

Some student have learned the interval notation for writing down ranges of values. These are acceptable if used correctly, but my preference is for the notation described above.

Not Preferred:

$$x \in (-\infty, 4]$$

Not Preferred:

$$x \in [0, 1] \cup [2, \infty)$$

As before, though, it is incorrect to include infinity in a closed interval since a variable can never be equal to infinity.

Incorrect:

$$x \in [0, 1] \cup [2, \infty]$$

6.7 Trigonometry

Problem:

Students get radians and degrees mixed up.

Solution:

Almost all problems involving angles in precalculus and calculus will use radians. Sometimes for "easy" angles, we might say "45 degrees" or "120 degrees" since these are easier to visualize than "$\pi/4$ radians" or "$2\pi/3$ radians". Nevertheless, the quicker we abonadon this practice and get used to radian measure, the easier life will be. Set your calculators to radian mode and never change them back!

In terms of notation, though, the only real issue to be settled is what to write. Since radians are the default unit in calculus, you don't need to specify anything extra if you want to state something in radians.

Correct:

$$\text{The angle is } \theta = \frac{3\pi}{4}.$$

If for any reason you do need degrees (and I can't think of many reasons you would), you must use the degree symbol.

Incorrect:

$$\text{The angle is } \theta = 135.$$

Correct:

$$\text{The angle is } \theta = 135^\circ.$$

Problem:

Trig functions have specialized notation that can lead to confusion.

Solution:

Learn the notational conventions for trig functions.

First of all, remember that trig functions are actually functions. That means that they require an argument.

Incorrect:

$$\sin$$

Preferred:

$$\sin x$$

or

Not Preferred:

$$\sin(x)$$

The former is more elegant than the latter since, really, the parentheses aren't necessary to clarify, but they are necessary in some situations.

If you want to take the cosine of $x + y$, then you can't use

Incorrect:

$$\cos x + y \text{ for "the cosine of } (x + y)\text{".}$$

Be careful to observe that I am not saying the expression $\cos x + y$ contains an error. I just mean that it is incorrect to use this if you want to take the cosine of $x + y$. Instead you need

Correct:

$$\cos(x + y) \text{ for "the cosine of } (x + y)\text{".}$$

If you write $\cos x + y$, what you're really saying is

Not Preferred:

$$(\cos x) + y,$$

but why not write this as

Preferred:

$$y + \cos x$$

if that's what you mean? That eliminates the ambiguity.

If the argument of the trig function contains two or more expressions multiplied together, parentheses are optional, but including them might be helpful for clarification.

Not Preferred:

$$\tan xyz$$

Preferred:

$$\tan(xyz)$$

Note that if you want $\tan x$ times yz, then is is *not* correct to write

Incorrect:

$$\tan xyz \text{ for "} \tan x \text{ times } yz\text{",}$$

nor is it correct to write

Incorrect:

$$\tan x(yz)$$

because it is not clear if the yz term is meant to be part of the tangent or not.

Rather, you need either

Not Preferred:

$$(\tan x)yz$$

or

Preferred:

$$yz \tan x.$$

See section 6.10, *Order of functions*, for more about this.

Exponentiating trig functions also has a special notation to help eliminate ambiguity. If you want to square the sine function, you cannot write

Incorrect:

$$\sin x^2 \text{ for "square the function} \sin x\text{"}.$$

What you mean is

Not Preferred:

$$(\sin x)^2,$$

but there is a standardized notational convention for this:

Preferred:

$$\sin^2 x.$$

The point here is that

Not Preferred:

$$\sin x^2 \text{ means "sine of } x^2\text{"},$$

so use

Preferred:

$$\sin (x^2)$$

instead to eliminate any ambiguity.

Also see section 7 on *Simplification of answers* for more on manipulating trig functions.

6.8 Logarithms

Problem:

Since the log functions—mostly we use the natural log function, $\ln x$—have their argument at the end like the trig functions, the same type of ambiguities can occur.

Solution:

The advice from section 6.7, *Trigonometry*, mostly applies here. Use parentheses when necessary to clarify, or better yet, leave the log function at the end of the expression. (But see section 6.10, *Order of functions*.)

The only thing about trig functions that doesn't apply to log functions is the special notation for exponentiating.

Incorrect:

$$\ln^2 x$$

Correct:

$$(\ln x)^2$$

Note that the expression

Not Preferred:

$$\ln x^2$$

means $\ln (x^2)$, so if you want that, you might want to eliminate any possibility of confusion, just like for trig functions.

Preferred:

$$\ln (x^2)$$

Nevertheless, it is up to you. Technically speaking, order of operations should leave this unambiguous even without the parentheses.

Also, be careful when using log rules since there are potential domain issues. It is not technically correct to say

Incorrect:

$$\ln (x^2) = 2 \ln x$$

since the expression on the left is valid for all $x \neq 0$, but the expression on the right is only defined for $x > 0$. So be sure to point out when there is a domain restriction imposed by simplifying:

Correct:

$$\ln (x^2) = 2 \ln x, \qquad x > 0.$$

See section 7 on *Simplification of answers*—especially section 7.4 on *Domain matching*—for more on this important aspect of manipulating log functions.

6.9 Inverse functions

Problem:

The standard notation for inverse functions is a bit unfortunate since it conflicts with standard exponential notation.

Solution:

The only inverse functions that commonly use the unfortunate exponential notation are the inverse trig functions, and even these have other names by which we can refer to them.

Not Preferred:

$$\sin^{-1} x$$

Preferred:

$$\arcsin x$$

If a function is just called $f(x)$, then there is no way around it.

Correct:

$$f^{-1}(x)$$

But note that this does not mean "$f(x)$ to the negative first power".

Incorrect:

$$f^{-1}(x) = \frac{1}{f(x)}$$

If we wish to write the reciprocal of $f(x)$, it is

Correct:

$$[f(x)]^{-1} = \frac{1}{f(x)}$$

Also be careful when differentiating an inverse function.

Incorrect:

$$(f^{-1})'(x) = (f')^{-1}(x)$$

The left-hand side of this equation is the derivative of the inverse function, whereas the right-hand side is the inverse of the derivative function. These are different, as shown by the example below.

If $f(x) = \tan x$, then $f'(x) = \sec^2 x$, so then we have

Correct:

$$f^{-1}(x) = \arctan x$$

Correct:

$$(f^{-1})'(x) = \frac{1}{1+x^2}$$

Correct:

$$(f')^{-1}(x) = \operatorname{arcsec}\sqrt{x}$$

6.10 Order of functions

Problem:

In an expression involving a product of more than one function, if the functions are written in the "wrong" order, the result can be ambiguous, or at best, ugly. There is no blue text here because this is really only a suggestion based on the way math is customarily written. But as long as the result in unambiguous, there isn't necessarily a right or a wrong way to order the terms in an expression.

Solution:

The recommendation is that you follow the order given below when constructing a complicated expression.

1. Constants that are numerals.
2. Constants designated by letters.
3. Single variables and powers of variables. (If more than one kind of letter is present, you will often see these in alphabetical order, although that is not a hard and fast rule.)
4. Exponentials.
5. Trig functions or log functions.
6. Roots.

For example,
Correct:

$$2\pi x^2 yze^{xy}\sin yz\sqrt{x^2+y^2+z^2}$$

Note that, in accordance with earlier advice, since the root comes last there can be no question as to what goes underneath it. Since the trig function is only followed by the radical, it is clear what the argument of sine is. The exponential function is clear because its argument is superscripted.

7 Simplification of answers

Problem:

Many professors require students to simplify their answers, but student often don't know the difference between actual simplification and other ways of rearranging mathematical expressions that aren't really "simplifications".

Solution:

The only general advice that can be given is that you study the following sections carefully to learn to simplify when necessary. (It's just as important to leave things alone when necessary!)

The "incorrect" examples in this section are not incorrect because they contain an error in notation. In fact, during intermediate steps of a calculation, expressions like this are likely to appear and they will be perfectly correct in that context. This section is only applicable to the form in which you present your final answer.

7.1 Redundant notation

Your final answer should not contain expressions like, for example, $0 \cdot x$, $1 \cdot x$, $x + 0$, $x + (-2)$, or $x - (-2)$.

Incorrect:

$$\frac{2x}{8y+12} = \frac{\overset{1}{\cancel{2}}x}{\overset{2}{\cancel{4}}(2y+3)} = \frac{1x}{2\,(2y+3)}$$

Correct:

$$\frac{2x}{8y+12} = \frac{\overset{1}{\cancel{2}}x}{\overset{2}{\cancel{4}}(2y+3)} = \frac{x}{2\,(2y+3)}$$

7.2 Factoring and expanding

It's important to note that there are certain manipulations that can change an expression without necessarily simplifying it. The question of factoring or expanding is usually not one of simplifying.

Correct:

$$4x^3 \ln x + 8x^2 (\ln x)^2 = 4x^2 \ln x\,(x + 2 \ln x)$$

The first expression is expanded and the second is factored, but the second expression is not really any more simple than the first. In fact, if we are doing calculus, it is usually *not* desirable to factor since doing so produces products. Recall that products are more difficult to differentiate and integrate than sums. The only reason we would want to factor is if we have a fraction and it is clear that by factoring, something will cancel from both the numerator and the denominator. On the other hand, if you are finding the roots of an equation, then you have no choice but to factor. The "correct" thing to do is more an issue of context than correct notation.

The same holds for

Correct:

$$x^3 + 2x^2 - x - 2 = (x - 1)(x + 1)(x + 2)$$

The expression on the left is easier to work with for purposes of doing calculus, and the expression on the right is easier to graph since it is immediately apparent where its roots are.

7.3 Basic algebra

Like terms should be added or subtracted together. In a fraction, common factors in the numerator and denominator should be canceled. All constants in any given term should be combined. Terms with different exponents and the same base should be combined as well.

Incorrect:

$$e^x + 2e^{2x} + 3e^x$$

Correct:

$$e^x + 2e^{2x} + 3e^x = 4e^x + 2e^{2x}$$

Incorrect:

$$\frac{5\sin x + 10\sin x\cos x}{25\sin x}$$

Correct:

$$\begin{aligned}\frac{5\sin x + 10\sin x\cos x}{25\sin x} &= \frac{\cancel{(5\sin x)}\,(1+2\cos x)}{\cancel{(5\sin x)}5}\\ &= \frac{1+2\cos x}{5}, \qquad x \neq k\pi \text{ for all integers } k\end{aligned}$$

(See section 7.4, *Domain matching*, for more about the condition

$$x \neq k\pi \text{ for all integers } k.)$$

Incorrect:

$$6xe^x(3x^2)e^y$$

Correct:

$$6xe^x(3x^2)e^y = 18x^3e^{x+y}$$

7.4 Domain matching

Problem:

When simplifying functions, students often ignore issues of domain.

Solution:

If the domain of the function on the left hand side of an equation is not the same as the domain of the function on the right hand side, then the functions are not really equal. So extra restrictions to the domain must be written explicitly so that the functions become equal.

Incorrect:

$$\frac{\ln x}{x \ln x} = \frac{1}{x}$$

The function on the left has as its domain $0 < x < 1$ and $x > 1$. (The natural log function has domain $x > 0$, but x cannot be 1 either since it would make the denominator zero.) On the other hand, the function on the right has as it's only restriction $x \neq 0$. So we must restrict the domain of $1/x$ if we want equality to hold.

Correct:

$$\frac{\ln x}{x \ln x} = \frac{1}{x}, \qquad 0 < x < 1 \text{ and } x > 1$$

But it is not necessary to identify the domain of everything you write down.

Not Preferred:

$$f(x) = \frac{1}{x}, \qquad x \neq 0$$

The $x \neq 0$ is redundant; every function comes with a domain and it is tacitly assumed that the function is only defined on that domain, even if it is not obvious from looking at the function what that domain is. This is different from the above example since in the above example we are trying to establish equality between two functions that appear to be algebraically equivalent.

Sometimes it is hard to spot when the domain needs to be restricted.

Correct:

$$\ln(e^x) = x$$

Incorrect:

$$e^{\ln x} = x$$

Correct:

$$e^{\ln x} = x, \qquad x > 0$$

In the first example, x can be anything since the domain of e^x is all real numbers. And since the range of e^x only consists of positive real numbers, there is no problem with the natural log function. So on the right, when we write x, we need not specify a domain since the range of the function x is also all real numbers.

But in the second example, the first function evaluated on the left is $\ln x$, which is only valid for $x > 0$. So this needs to be specified on the right for the function x, as in the third example above.

Here is an interesting example from section 5.2 above.

Correct:

$$\frac{x^2 - 2x + 1}{x^2 - 1} = \frac{(x-1)\cancel{(x-1)}}{(x+1)\cancel{(x-1)}}$$
$$= \frac{x-1}{x+1}, \qquad x \neq 1$$

Why isn't the domain listed as $x \neq \pm 1$? While it is true that the domain of the function on the left is $x \neq \pm 1$, the domain of the function on the right is already $x \neq -1$. So this doesn't need to be stated to make the two functions equal. The only difference between the two functions—and therefore the only restriction that needs to be mentioned—is $x \neq 1$.

Another situation where the domain matters is when working with inverse functions. (See section 6.9 for more information.) For example, the processes of squaring and taking a square root are inverses of each other. Or are they?

Incorrect:

$$\sqrt{x^2} = x$$

The problem is best illustrated plugging in a number. If $x = -2$ then $\sqrt{(-2)^2} = \sqrt{4} = 2$, not -2. The function x^2 is not one-to-one, and so there are generally two values of x that give the same value of x^2. That is, unless you restrict the domain.

Correct:

$$\sqrt{x^2} = x, \qquad x \geq 0$$

An even better way to fix this problem is to write

Correct:

$$\sqrt{x^2} = |x|$$

for now the domains on the left and the right agree and consist of all real numbers. The fact that the answer is always positive is taken care of by the absolute value signs.

This is not exactly the difficulty with

Incorrect:

$$\left(\sqrt{x}\right)^2 = x$$

Just having the term $\sqrt{x}$ in the expression automatically forbids us from plugging in a negative value of x. Then squaring a positive number still gives a positive number, so in this case the inverse process works perfectly. In other words, for any given positive value of x as input, that same positive value is the output. The problem here, then, is the same problem first described in this section, which is that the domains on the left and right don't agree.

Correct:

$$\left(\sqrt{x}\right)^2 = x, \qquad x \geq 0$$

7.5 Using identities

It is not always expected that you will use more complicated identities to change expressions into other expression that might be simplified. For example, reasonable professors will accept

Correct:

$$\frac{2\sin x + \sin 2x}{2\sin x}$$

even though

Correct:

$$\begin{aligned}\frac{2\sin x + \sin 2x}{2\sin x} &= \frac{2\sin x + 2\sin x\cos x}{2\sin x}\\ &= \frac{\cancel{(2\sin x)}(1+\cos x)}{\cancel{2\sin x}}\\ &= 1+\cos x, \qquad x \neq k\pi \text{ for all integers } k\end{aligned}$$

using the trig identity $\sin 2x = 2\sin x\cos x$.

(Although it is not a notational error, so I won't dwell on it here, notice that it is a mathematical error to cancel the $2\sin x$ before factoring it out of both terms in the numerator.

Incorrect:

$$\frac{\cancel{2\sin x} + \sin 2x}{\cancel{2\sin x}} = 1 + \sin 2x$$

Just remember that you can't cancel terms until everything in the numerator and denominator is written as a product of factors.)

7.6 Log functions and exponential functions

Simplifying logarithms also depends on context. One can use log rules to manipulate expressions without necessarily simplifying them.

Correct:

$$\ln x + \ln \sin x = \ln (x \sin x)$$

Both sides of this equation are correct. If I am differentiating, I would rather use the sum on the left. But the expression on the right is a bit more compact than the one on the left.

It would probably not be preferable to leave an expression of the form

Not Preferred:

$$\ln x + \ln \left(x^2\right)$$

unsimplified, since

Preferred:

$$\ln x + \ln \left(x^2\right) = \ln x + 2 \ln x = 3 \ln x$$

or also

Preferred:

$$\ln x + \ln \left(x^2\right) = \ln \left(x \cdot x^2\right) = \ln \left(x^3\right)$$

depending on which log rule you decide to use. Of course, $3 \ln x = \ln (x^3)$ and neither of these forms is particularly more desirable than the other.

When logs and exponentials appear together, there is usually some simplification that should be done. This is because the log function and the exponential function are inverse functions. One operation cancels the other, just like addition and subtraction, or multiplication and division.

Incorrect:

$$e^{\ln (3x)}$$

Correct:

$$e^{\ln (3x)} = 3x, \qquad x > 0$$

Note that this may involve the use of an intermediate log rule.

Incorrect:

$$e^{2 \ln x}$$

Correct:

$$e^{2 \ln x} = e^{\ln \left(x^2\right)} = x^2, \qquad x > 0$$

Also

Incorrect:

$$\ln e^{\sin x}$$

Correct:

$$\ln e^{\sin x} = \sin x$$

7.7 Trig functions and inverse trig functions

As before, when functions and their inverse functions appear in the same expression, there is often the need to cancel.

Incorrect:

$$\tan(\arctan x)$$

Correct:

$$\tan(\arctan x) = x$$

Be careful, though! Strictly speaking, the tangent and the arctangent functions are not quite inverses.

Incorrect:

$$\arctan(\tan x) = x$$

The tangent function is not one-to-one, so, for example

$$\tan\left(\frac{9\pi}{4}\right) = 1$$

but

$$\arctan 1 = \frac{\pi}{4}$$

So taking the tangent first and then taking arctangent doesn't always give us back the original number. To correct this we have to ensure that the domain of the function is restricted.

Correct:

$$\arctan(\tan x) = x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(See section 7.4, *Domain matching*, for more about the condition

$$-\frac{\pi}{2} < x < \frac{\pi}{2}.)$$

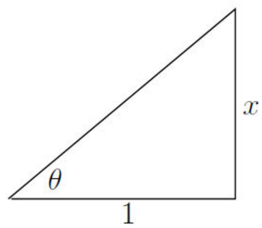
The situation is the same with other inverse trig functions. Your textbook should have a section that contains more information on the domains of these functions.

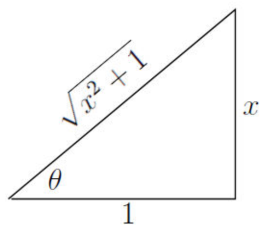
Another type of problem occurs with expressions like, for example,

Incorrect:

$$\sin(\arctan x)$$

The trick to simplifying this is to see that $\arctan x$ describes the angle θ in the figure. (This is the same as saying that $\tan\theta = \frac{x}{1} = x$.)





So $\sin(\arctan x)$ is $\sin\theta$. What is $\sin\theta$? Use the Pythagorean Theorem to fill in the missing side (the hypotenuse) and then read off the sine as opposite over adjacent.

Therefore,

Correct:

$$\sin(\arctan x) = \sin\theta = \frac{x}{\sqrt{x^2+1}}$$

Here, also, one must consider the domain for which these functions are defined and one-to-one. (Although in this case there isn't anything extra to say since the domain of the expression on the left is all real numbers as is the domain of the expression on the right.)

8 Limits

8.1 Limit notation

Problem:

The biggest problem that students have with limit notation is just remembering to use it. Often, the limit will appear in the first step of the problem and then disappear in every expression that follows.

Solution:

Continue writing lim in front of every expression that requires it until you are ready to take the limit. Since this usually occurs in the last step of most problems, you can see why this advice is so important. Alternatively, after taking the limit, the lim must disappear from the expression.

Incorrect:

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \frac{\Delta x\,(2x + \Delta x)}{\Delta x} \\ &= 2x + \Delta x \\ &= 2x\end{aligned}$$

Incorrect:

$$\begin{aligned}\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x\,(2x + \Delta x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2x + \Delta x \\ &= \lim_{\Delta x \to 0} 2x \\ &= 2x\end{aligned}$$

In the second incorrect example above, the next to last line should not have a limit in front of it since the limit here has been taken already.

Correct:

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x\,(2x + \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x) \\
&= 2x
\end{aligned}$$

Also pay attention to the next to last line here. In this correct example, we have placed parentheses around $2x + \Delta x$. In the steps above this, parentheses were not necessary since it was clear that the limit applied to the whole fraction, but when the fraction goes away, we are left with a sum. Although it is intuitively clear what we mean here, without the parentheses this would technically be

$$\left(\lim_{\Delta x \to 0} 2x\right) + \Delta x$$

So without the parentheses, the limit would only apply to the $2x$ term. The lesson here is this: when a sum or difference is involved in a limit, you must put parentheses around the expression so that it is clear that the limit applies to all terms in the expression. This is not necessary for a product or quotient.

8.2 Infinite limits

Problem

In infinite limits, the symbol for infinity, ∞, is used too casually.

[Do something with this:]

An inequality like

Correct:

$$\lim_{x\to\infty} \frac{2x}{3x-5} < \infty$$

is correct and is typically interpreted to mean that the limit is a finite number.

Solution:

Infinity is not a number. This is precisely why limits need to be used when dealing with quantities that tend to infinity. One must not substitute the symbol ∞ for a variable as one would plug in a real number.

Incorrect:

$$\lim_{x\to\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

There is no need for this intermediate step. It is shown in any first semester calculus course what the value of this limit is. (And of course it is intuitively clear that as the denominator gets bigger, the fraction gets smaller, so that in the limit, the expression goes to zero.)

Correct:

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

In the event that the limit does not exist because it grows without bound, it is permissible to use the symbol ∞ for your final answer. In fact, it is preferable because it is more enlightening than just saying that the limit does not exist. Increasing without bound to infinity or decreasing without bound to negative infinity are much more controlled types of behaviors than limits that do not exist because they oscillate wildly or have some other erratic behavior.

Correct:

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2+1}{x} &= \lim_{x\to\infty}\frac{\frac{x^2}{x}+\frac{1}{x}}{\frac{x}{x}}\\ &= \lim_{x\to\infty}\left(x+\frac{1}{x}\right)\\ &= \infty\end{aligned}$$

(Here we have used the standard trick of dividing both the numerator and the denominator by the highest power of x found in the denominator. Of course, in this case, we didn't really need to do this since we could have obtained the same answer just by separating the expression into two fractions. Nevertheless, if there is more than just a single term in the denominator, the fraction cannot be separated, so it is a good idea to remember the standard trick.)

9 Derivatives

9.1 Derivative notation

Problem:

There are several different ways or writing a derivative. Some are more appropriate than others depending on the context. Also, each convention has its own notational pitfalls.

Solution:

Learn the notational conventions for all the different ways of writing derivatives.

9.1.1 Lagrange's notation

When a function is given, such as $f(x) = x^5$, then use the "prime" notation, sometimes known as Lagrange's notation.

Correct:

$$f'(x) = 5x^4$$

This function can also be written as $y = x^5$ or $y(x) = x^5$ if you want to emphasize that x is the independent variable. (One might write a function using x and y when one wants to think of the function as a graph, since using x and y suggests Cartesian coordinates.) In this case, the prime notation is

Correct:

$$y' = 5x^4$$

or

$$y'(x) = 5x^4$$

For second or third derivatives, use two or three primes.

Correct:

$$f''(x) = 20x^3$$
$$f'''(x) = 60x^2$$

Correct:

$$y'' = 20x^3$$
$$y''' = 60x^2$$

But for fourth derivatives and higher, using primes would be unwieldy.

Incorrect:

$$f''''(x) = 120x$$

Instead use the notation

Correct:

$$f^{(4)}(x) = 120x$$
$$y^{(5)} = 120$$

Be sure to put the number of derivative in parentheses. Otherwise, it will look like an exponent.

Incorrect:

$$f^5(x) = 120$$

9.1.2 Leibniz's notation

Another way of writing derivatives is using Leibniz notation. I generally do not tend to use Leibniz notation, but before explaining why, let's see how it works. For a function such as $g(x) = \sin 2x$, the first derivative would be written as

Correct:

$$\frac{dg}{dx} = 2\cos 2x$$

For a function written as $y = \sin 2x$, the Leibniz notation for the derivative is

Correct:

$$\frac{dy}{dx} = 2\cos 2x$$

The Leibniz notation for higher derivatives is especially unfortunate, but the following are technically correct.

Correct:

$$\frac{d^2g}{dx^2} = -4\sin 2x$$

Correct:

$$\frac{d^2y}{dx^2} = -4\sin 2x$$

Correct:

$$\frac{d^3g}{dx^3} = -8\cos 2x$$

Correct:

$$\frac{d^3y}{dx^3} = -8\cos 2x$$

Using Leibniz notation, one can also view derivatives as operators; that is, the derivative is applied to a function and produces another function. For example, if $h(x) = \ln x$, then taking the derivative can be written as

Correct:

$$\frac{d}{dx}[h(x)] = \frac{d}{dx}(\ln x) = \frac{1}{x}$$

In other words, the operator d/dx takes as its argument the function $\ln x$ and produces the function $1/x$.

For the function written as $y = \ln x$, we can write

Correct:

$$\frac{d}{dx}(y) = \frac{d}{dx}(\ln x) = \frac{1}{x}$$

Of course,

$$\frac{d}{dx}(y) = \frac{dy}{dx}$$

This operator idea helps explain the bizarre notation for higher derivatives. To get the second derivative, we must take the derivative of the first derivative:

Correct:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Now it is a mistake to think of the letter "*d*" as if it were a variable. It is also mathematically incorrect to consider "*dy*" or "*dx*" as actual quantities. The "quotient" $\frac{dy}{dx}$ is not really a quotient, although it is intuitively suggestive of the idea of calculating a slope as "rise over run". So $\frac{d}{dx}$ is an indivisible piece of notation, as is $\frac{dy}{dx}$. Nevertheless, if we ignore the mathematical reality and treat these symbols as if they were all variables that could be manipulated, we see that the above expression "multiplies" to give

Correct:

$$\frac{d^2y}{dx^2}$$

Given that we accept the somewhat egregious notation $\frac{dy}{dx}$ as shorthand for the derivative, it is no more or less egregious to use $\frac{d^2y}{dx^2}$ as shorthand for the second derivative. We simply must remember that these are not fractions and they cannot be manipulated like fractions. (But see section 9.2 below on the *Chain rule*.)

Now there are very few situations where the Leibniz notation has any advantage over the prime notation. The one exception is if you are asked to take the derivative of a function that is not assigned to a variable. For example, if the problem says, "Find the derivative of $\tan x$," you cannot use a prime because there is no letter to which you can attach it. Rarely, you may see

Not Preferred:

$$[\tan x]' = \sec^2 x$$

but it's better to write

Preferred:

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

Of course, if the problem says, "Find the derivative of $f(x) = \tan x$," then the answer is easiest expressed with primes.

Correct:

$$f'(x) = \sec^2 x$$

So in the derivative tables inside the front or back covers of your books, you will see Leibniz notation since these functions appear by themselves and are not assigned to another variable.

But Leibniz notation is very awkward when you have to evaluate a derivative at a specific point. If we want to find the slope of the tangent line to the curve $y = x^3$ at the point $x = 2$, one would first find the derivative $\frac{dy}{dx} = 3x^2$ and then write

Not Preferred:

$$\left.\frac{dy}{dx}\right|_{x=2} = 3(2)^2 = 12$$

The prime notation is much easier since it already looks like a function. The derivative function is $y'(x) = 3x^2$, so plugging in $x = 2$ works the same way as it would for any other function.

Preferred:

$$y'(2) = 3(2)^2 = 12$$

9.1.3 Euler's notation

This notation is rather uncommon in basic calculus, so we won't dwell on it unnecessarily.

The following shows the derivative as an operator using a capital D with a subscript indicating the independent variable.

Not Preferred:

$$D_x\left[\arctan x\right] = \frac{1}{1+x^2}$$

The reason this is not preferred is that mathematicians usually use subscripts like this to indicate "partial" derivatives in multi-variable calculus. Higher derivatives can be written using superscripted numerals.

Not Preferred:

$$D_x^2\left[\arctan x\right] = \frac{-2x}{\left(1+x^2\right)^2}$$

9.1.4 Newton's notation

Newton's notation uses dots above the name of the function. Like Euler's notation, it is rare to see this in basic calculus. It is more commonly seen in physics books and often refers to derivatives with respect to time. For example, if $x(t) = t^2 + 4t + 8$, then

Not Preferred:

$$\dot{x}(t) = 2t + 4$$

Not Preferred:

$$\ddot{x}(t) = 2$$

There is no good way to extend this notation to third derivatives or higher. (In basic physics problems, though, this is rarely necessary since one and two dots are enough to express velocity and acceleration respectively.)

9.1.5 Other notation issues

A function is rarely equal to its derivative. (It is true only for the function e^x.) Nevertheless, many students use an equal sign to go from one step to the next in taking derivatives. When calculating a derivative, indicate this on a new line and be sure not to put an equal sign where it does not belong.

These are just a few of the ways this might be done incorrectly.

Incorrect:

$$f(x) = \sqrt{x} = x^{1/2} = (1/2)x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

Incorrect:

$$\begin{aligned} f(x) = \sqrt{x} &= x^{1/2} \\ &= (1/2)x^{-1/2} \\ &= \frac{1}{2x^{1/2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Incorrect:

$$\begin{aligned} f(x) &= \sqrt{x} = x^{1/2} \\ (1/2)x^{-1/2} &= \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

Incorrect:

$$\begin{aligned} f(x) &= \sqrt{x} = x^{1/2} \\ \longrightarrow (1/2)x^{-1/2} &= \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

Correct:

$$\begin{aligned} f(x) &= \sqrt{x} = x^{1/2} \\ f'(x) &= (1/2)x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

9.2 Chain rule

Problem:

If students make mistakes involving the chain rule, they are usually mathematical errors and not notational errors. But talking about the chain rule using Leibniz notation may engender some confusion.

Solution:

Recall from section 9.1.2 that the expression $\frac{dy}{dx}$ using Leibniz notation is not really a fraction. That is, it is a mistake to think about dy and dx as if they were actual quantities in the numerator and denominator. The chain rule in Leibniz notation says that if y is a function of u and u is a function of x, then the derivative of y with respect to x is given by

Correct:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

In other words, one can think of $y(u)$ as the "main function" while $u(x)$ is the "inside function". So one has to differentiate $y(u)$ first and then multiply this by the derivative of the inside function $u(x)$.

But this notation makes it look like the chain rule works because of "cancellation".

Incorrect:

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}}\frac{\cancel{du}}{dx}$$

The prime notation avoids this problem. In Lagrange's notation the chain rule is

Correct:

$$\frac{d}{dx}\left[\, y(u(x))\right] = y'(u(x))u'(x)$$

10 Integrals

10.1 Integral notation

Problem:

Students often neglect the importance of the dx.

Solution:

The dx is a critical component of the integral and must always be included and used correctly. As a Riemann sum, the integral is approximated by rectangles whose widths are given by Δx. In the limit, these rectangles are considered to be "infinitesimally small" and the Δx becomes a dx. Students are particularly displeased if they lose points for forgetting the dx since even without it, it is usually clear what the integral is. While this may be true, it is not an excuse for misusing notation. In English we are required to use correct grammar and spelling, even when it might be "clear" what we mean without it. Mathematics is no different.

Incorrect:

$$\int xe^{x^2} = \frac{1}{2}e^{x^2} + C$$

Correct:

$$\int xe^{x^2}\,dx = \frac{1}{2}e^{x^2} + C$$

Sometimes you will separate an integral into two separate integrals. Each piece must, of course, have its own dx.

Incorrect:

$$\int_{-1}^{1} \left(e^x + e^{2x}\right)\,dx = \int_{-1}^{1} e^x + \int_{-1}^{1} e^{2x}$$

Incorrect:

$$\int_{-1}^{1} \left(e^x + e^{2x}\right)\,dx = \int_{-1}^{1} e^x + \int_{-1}^{1} e^{2x}\,dx$$

Correct:

$$\int_{-1}^{1} \left(e^x + e^{2x}\right)\,dx = \int_{-1}^{1} e^x\,dx + \int_{-1}^{1} e^{2x}\,dx$$

Another problem with dx is making sure it is attached to the whole integrand. If the integrand is a sum or difference, parentheses are necessary.

Incorrect:

$$\int \sin x + \cos x \, dx = -\cos x + \sin x + C$$

Correct:

$$\int (\sin x + \cos x) \, dx = -\cos x + \sin x + C$$

Finally, there must only be one variable in the integrand. (But see section 10.4 on *Integration by substitution* for more on this.) This problem is caused mostly by carelessness stemming from the habit of always writing dx even though the variable could be anything.

Incorrect:

$$\int \sqrt{t} \, dx$$

Correct:

$$\int \sqrt{t} \, dt$$

10.2 Definite integrals

Problem:

The definite integral and the indefinite integral are two different objects. The former is a number and the latter is a function. (Technically, an indefinite integral represents an infinite collection of functions differing from each other by a constant. See section 10.3 on *Indefinite integrals* below.) Much notational confusion results from misunderstanding the differences between these two.

Solution:

The definite integral is a number, so the answer should be a number, not a function.

Incorrect:

$$\int_2^4 \frac{1}{x \ln x}\, dx = \ln(\ln x)$$

Incorrect:

$$\int_2^4 \frac{1}{x \ln x}\, dx = \ln(\ln x) + C$$

Correct:

$$\int_2^4 \frac{1}{x \ln x}\, dx = \Big[\ln(\ln x)\Big]_2^4 = \ln(\ln 4) - \ln(\ln 2)$$

Pay attention to the notation for the limits of integration. There are several ways to do it.

Preferred:

$$\Big[\ln(\ln x)\Big]_2^4$$

as in the earlier example. Or

Not Preferred:

$$\ln(\ln x)\Big]_2^4$$

Not Preferred:

$$\ln(\ln x)\Big|_2^4$$

The reason the first of the above expressions is better than the latter two is that it clearly indicates the function for which we are to substitute our limits of integration. There is the possibility of ambiguity when the function is not enclosed in brackets.

Incorrect:

$$\sin x + \cos x \Big]_{\pi/4}^{\pi/2}$$

It looks like we are only using the limits of integration on the function $\cos x$. So enclose the whole thing in brackets.

Correct:

$$\Big[\sin x + \cos x\Big]_{\pi/4}^{\pi/2}$$

10.3 Indefinite integrals

Problem:

The biggest problem is remembering to add $+C$ when integrating.

Solution:

The indefinite integral is an antiderivative. Since the derivative of a constant is zero, any constant can be added to a function and the derivative of that function won't change. Therefore, given a function, there are an infinite number of functions whose derivative is the given function, all differing from one another by a constant. The result of integration must contain $+C$.

Incorrect:

$$\int e^x \cos(e^x)\,dx = \sin(e^x)$$

Correct:

$$\int e^x \cos(e^x)\,dx = \sin(e^x) + C$$

You must add the $+C$ the moment you integrate. It is not enough just to tack it onto the last step. As this error occurs most frequently when doing u-substitution, we will discuss it section 10.4 which follows.

If you have more than one integral in the expression, I suppose it's technically correct to include a constant of integration for each integral. Nevertheless, your answer will not be simplified if you leave it this way.

Incorrect:

$$\int x\,dx + \int x^2\,dx = \frac{1}{2}x^2 + C_1 + \frac{1}{3}x^3 + C_2$$

This is quite pointless since the sum of two arbitrary constants is another arbitrary constant, so we just subsume all constants into one.

Correct:

$$\int x\,dx + \int x^2\,dx = \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$$

10.4 Integration by substitution

Problem:

Integration by substitution is often called u-substitution. There are many possible problems with the u-substitution procedure.

Solution:

First, one must be sure that every piece of the original integrand is changed to reflect the new choice of variable. The most common mistake here is forgetting to use differential notation correctly. For example, if we are trying to integrate

$$\int \frac{\cos x}{1+\sin^2 x}\,dx$$

then the correct u-substitution is

Correct:

$$u = \sin x$$
$$du = \cos x\,dx$$

But it is incorrect to write

Incorrect:

$$u = \sin x$$
$$du = \cos x$$

because there is no dx in the du term.

Before we talk about possible mistakes, let's see the correct way to complete this problem.

Correct:

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{du}{1+u^2} \quad (1)$$
$$= \arctan u + C \quad (2)$$
$$= \arctan(\sin x) + C \quad (3)$$

Every line of this correct method contains a potential error to avoid. The first step, line (1), shows that a correct u-substitution must replace all the x's in the original integrand with u's.

Incorrect:

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{\cos x\,dx}{1+u^2}$$

Incorrect:

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{dx}{1+u^2}$$

Incorrect:

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{\cos x\,du}{1+u^2}$$

There is only one place where it is permissible to see u's and x's in the same integral, and to be legitimate, all the x's must cancel in the subsequent step. Recall that one way to do u-substitution is to solve the equation $du = \cos x\,dx$ for dx.

$$dx = \frac{du}{\cos x}$$

and then plug this in, obtaining

Correct:

$$\int \frac{\cos x}{1+\sin^2 x}\,dx = \int \frac{\cancel{\cos x}}{1+u^2}\frac{du}{\cancel{\cos x}} = \int \frac{du}{1+u^2}$$

The fact that all the x's cancel means that the technique worked.

Line (2) above shows that the $+C$ must be written the moment you integrate.

Incorrect:

$$\begin{aligned}\int \frac{\cos x}{1+\sin^2 x}\,dx &= \int \frac{du}{1+u^2}\\ &= \arctan u\\ &= \arctan(\sin x) + C\end{aligned}$$

It's not enough just to tack it onto your answer at the end. If you ever take differential equations, you'll see examples where the answers changes drastically if you wait to include the constant.

Finally, line (3) shows that you must change the u's back to x's for the final answer.

(Note that, unlike the expressions described in section 7.7 on *Trig functions and inverse trig functions*, there is no obvious way to modify the expression $\arctan(\sin x)$.)

Extra care must be taken when performing u-substitution with definite integrals. Suppose we are trying to calculate

$$\int_1^3 \frac{x}{1+x^2}\,dx$$

The correct u-substitution here is

$$u = 1 + x^2$$
$$du = 2x\, dx$$
$$dx = \frac{du}{2x}$$

but it would be wrong to proceed as follows.

Incorrect:

$$\int_1^3 \frac{x}{1+x^2}\, dx = \int_1^3 \frac{\cancel{x}}{u}\left(\frac{du}{2\cancel{x}}\right) = \frac{1}{2}\int_1^3 \frac{du}{u}$$
$$= \frac{1}{2}\Big[\ln u\Big]_1^3 = \frac{1}{2}\left(\ln 3 - \cancelto{0}{\ln 1}\right)$$
$$= \frac{\ln 3}{2}$$

The variable changed from x to u correctly, but the limits of integration did not change accordingly, so the final answer is completely wrong.

Although the next variant gives the correct answer, it is still notationally wrong.

Incorrect:

$$\int_1^3 \frac{x}{1+x^2}\, dx = \int_1^3 \frac{\cancel{x}}{u}\left(\frac{du}{2\cancel{x}}\right) = \frac{1}{2}\int_1^3 \frac{du}{u}$$
$$= \frac{1}{2}\Big[\ln u\Big]_1^3 = \frac{1}{2}\Big[\ln\left(1+x^2\right)\Big]_1^3$$
$$= \frac{1}{2}(\ln 10 - \ln 2) = \frac{1}{2}\left[\ln\left(\frac{10}{2}\right)\right]$$
$$= \frac{\ln 5}{2}$$

If x goes between 1 and 3 in the original integral, then u must go from 2 to 10. (If $x = 1$, then $u = 1 + (1)^2 = 2$, and if $x = 3$, then $u = 1 + (3)^2 = 10$.)

Correct:

$$\int_1^3 \frac{x}{1+x^2}\, dx = \int_2^{10} \frac{\cancel{x}}{u}\left(\frac{du}{2\cancel{x}}\right) = \frac{1}{2}\int_2^{10} \frac{du}{u}$$
$$= \frac{1}{2}\Big[\ln u\Big]_2^{10} = \frac{1}{2}(\ln 10 - \ln 2) = \frac{1}{2}\left[\ln\left(\frac{10}{2}\right)\right]$$
$$= \frac{\ln 5}{2}$$

As soon as u took over for x, the limits of integration changed to reflect this. Note that the following is correct, but is way more work than you need to do.

Not Preferred:

$$\begin{aligned}
\int_1^3 \frac{x}{1+x^2}\,dx &= \int_2^{10} \frac{\cancel{x}}{u}\left(\frac{du}{2\cancel{x}}\right) = \frac{1}{2}\int_2^{10} \frac{du}{u} \\
&= \frac{1}{2}\Big[\ln u\Big]_2^{10} = \frac{1}{2}\Big[\ln\left(1+x^2\right)\Big]_1^3 \\
&= \frac{1}{2}\left(\ln 10 - \ln 2\right) = \frac{1}{2}\left[\ln\left(\frac{10}{2}\right)\right] \\
&= \frac{\ln 5}{2}
\end{aligned}$$

It is correct because when the integral uses u, the corresponding u-limits appear, and when it switches back to x, the original limits are restored. However, once you have converted to u and figured out the new limits of integration, you might as well forget about the original x's and just use the new variable. The new function will generally be easier to manipulate anyway—that's the whole point of u-substitution!

One way to do u-substitution with definite integrals is to avoid the issue altogether by first solving an indefinite integral. You may want to use this method if the new limits of integration corresponding to u are ugly. Consider the integral

$$\int_{-2}^{2} \frac{\arctan x}{1+x^2}\,dx$$

The correct substitution here is

$$\begin{aligned}
u &= \arctan x \\
du &= \frac{dx}{1+x^2}
\end{aligned}$$

In the case, though, the new limits would be $u = \arctan(-2)$ and $u = \arctan 2$. Even if it were okay to use calculators to evaluate these new limits (see section 4 on *Use of calculators*) it would be very ugly to try to carry these new limits through the rest of the problem and then try to evaluate the integral at the end using these limits. So instead, solve the problem

$$\int \frac{\arctan x}{1+x^2}\,dx$$

first.

Correct:

$$\begin{aligned}\int \frac{\arctan x}{1+x^2}\,dx &= \int \arctan x \left(\frac{dx}{1+x^2}\right)\\ &= \int u\,du\\ &= \frac{u^2}{2}+C\\ &= \frac{(\arctan x)^2}{2}+C\end{aligned}$$

and then take this answer and evaluate it with the original x limits.

$$\begin{aligned}\left[\frac{(\arctan x)^2}{2}\right]_{-2}^{2} &= \frac{(\arctan 2)^2}{2} - \frac{(\arctan(-2))^2}{2}\\ &= \frac{(\arctan 2)^2-(\arctan(-2))^2}{2}\end{aligned}$$

It's extremely important that these two parts of this process not be on the same line connected by an equal sign.

Incorrect:

$$\frac{(\arctan x)^2}{2}+C = \left[\frac{(\arctan x)^2}{2}\right]_{-2}^{2}$$

Remember, the indefinite integral is a function (or collection of functions if you count the $+C$) and the definite integral is a number. The two cannot be equal. Think of the indefinite integral as an intermediate computation you do off to the side before you go back to the main problem and plug in the limits to get the numerical answer.

10.5 Improper integrals

Problem:

Because improper integrals involve the notion of infinity, the same kinds of misunderstandings occur as with infinite limits.

Solution:

Remember that infinity is not a number. It may appear as a limit of integration, but this is only shorthand. To do any calculations, you must immediately turn any infinities into limits where the tools of calculus can be applied properly.

Incorrect:

$$\int_1^\infty e^{-x}\,dx = \Big[-e^{-x}\Big]_1^\infty$$

Correct:

$$\int_1^\infty e^{-x}\,dx = \lim_{t\to\infty}\int_1^t e^{-x}\,dx = \lim_{t\to\infty}\Big[-e^{-x}\Big]_1^t$$

Even when infinity is not one of the limits of integration, there still may be a need to use limits. The following integral looks deceptively simple.

Incorrect:

$$\int_0^4 \frac{1}{x}\,dx = \left[\frac{-1}{x^2}\right]_0^4$$

Of course, if you tried to continue, you would immediately see why there's a problem. Neither the original integrand, nor the function that results from integrating, are defined for $x = 0$. Instead we need

Correct:

$$\int_0^4 \frac{1}{x}\,dx = \lim_{t\to 0^+}\int_t^4 \frac{1}{x}\,dx = \lim_{t\to 0^+}\left[\frac{-1}{x^2}\right]_t^4$$

11 Sequences and series

11.1 Sequences

Problem:

Sequences are infinite sets of numbers, so as always when dealing with infinity, some care must be taken.

Solution:

Here are the basics of sequence notation.

First, be careful when using "dots" to indicate a pattern.

Incorrect:

$$\left\{\frac{1}{2}, \frac{1}{4}, \dots\right\}$$

What is the pattern here? Is the next term 1/6, each denominator being a multiple of two? Or is it 1/8, with each denominator being twice the last?

If we include a few more terms,

Not Preferred:

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\}$$

then it is fairly clear what the pattern is, but it is better to be completely explicit about it.

Preferred:

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots, \frac{1}{2^n}, \dots\right\}$$

Notice that there are dots before the term $\frac{1}{2^n}$ since there are missing terms up to that point, and there are dots after it as well. These dots must be there, for without them, the sequence would be finite.

Actually, neither of these is really preferred since they are both a bit unwieldy. As long as you are indicating that the nth term is $\frac{1}{2^n}$, you might as well use

Correct:

$$\left\{\frac{1}{2^n}\right\}$$

or even better,

Correct:

$$\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$$

which are far more concise.

If you want to use a variable to identify the nth term, it is written like

Correct:

$$a_n = \frac{1}{2^n}$$

(or b_n or c_n or any other letter, although letters toward the beginning of the alphabet are traditional).

But it is incorrect to write either of the following.

Incorrect:

$$\{a_n\} = \frac{1}{2^n}$$

Incorrect:

$$a_n = \left\{\frac{1}{2^n}\right\}$$

since each a_n only stands for one number, the nth term. Instead, this should be

Correct:

$$\{a_n\} = \left\{\frac{1}{2^n}\right\}$$

or even

Correct:

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$$

The latter is a bit bulkier, so if we don't care with which term the sequence starts, then the former is preferred. For example, when computing limits we know that any finite number of terms can be changed without affecting the limit, so it becomes immaterial which value of n initiates the sequence.

By now, it should come as no surprise that the following is incorrect.

Incorrect:

$$\lim_{n\to\infty} \{a_n\} = 0$$

This should be

Correct:

$$\lim_{n\to\infty} a_n = 0$$

11.2 Series

Problem:

The same types of problems occur for series as with sequences. Again, these problems often involve difficulties with infinity.

Solution:

Just as with sequences, be careful about using dots to indicate a pattern in a series.

Incorrect:

$$\frac{1}{2} + \frac{1}{4} + \ldots$$

Not Preferred:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Preferred:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \ldots$$

As before, these dots must be present or else the sum would not be an infinite sum.

The more correct way to write this is using sigma notation.

Correct:

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

If you're using a variable for the sequence of terms comprising your series, then the notation is

Correct:

$$\sum_{n=1}^{\infty} a_n$$

If you are manipulating a sum using sigma notation, you must write the sigma before each step, and each sigma must continue to carry bounds for the index. (Here, n is the index, going from one to infinity.)

Incorrect:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^{n+1}$$
$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n = \frac{1/2}{1-1/2}$$
$$= 1$$

Correct:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1}$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^n = \frac{1/2}{1-1/2}$$
$$= 1$$

The sum dropped out in the second to the last expression because here we are using a formula to evaluate this geometric series.

If each summand itself consists of a sum or difference of terms, then parentheses must be used so that the infinite sum is applied to each term.

Incorrect:

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

Correct:

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$